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Localization and traces in open-closed topological Landau-Ginzburg models

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Abstract: We reconsider the issue of localization in open-closed B-twisted Landau-Ginzburg models with arbitrary Calabi-Yau target. Through careful analsysis of zero-mode reduction, we show that the closed model allows for a one-parameter family of localization pictures, which generalize the standard residue representation. The parameter λ which indexes these pictures measures the area of worldsheets with S^2 topology, with the residue representation obtained in the limit of small area. In the boundary sector, we find a double family of such pictures, depending on parameters λ and μ which measure the area and boundary length of worldsheets with disk topology. We show that setting $\mu = 0$ and varying λ interpolates between the localization picture of the B-model with a noncompact target space and a certain residue representation proposed recently. This gives a complete derivation of the boundary residue formula, starting from the explicit construction of the boundary coupling. We also show that the various localization pictures are related by a semigroup of homotopy equivalences.

Keywords: Topological Field Theories, Topological Strings, D-branes.

Contents

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1. Introduction

Closed topological Landau-Ginzburg models [1] have been a useful testing ground for string theory. They make direct contact with the topological sector of rational conformal field theories through the Landau-Ginzburg approach to minimal models, and arise as phases of $N = 2$ string compactifications [2]. Moreover, they give explicit realizations of the WDVV equations and examples of Frobenius manifolds, and have interesting relations with singularity theory.

In a similar vein, one can expect to learn important lessons about open string theory by studying topological Landau-Ginzburg models in the presence of D-branes (see [30, 11] for some recent results in this direction). While basic D-brane constructions were considered by many authors (see [3] and references therein), a systematic study has been hampered by the lack of a reasonably general description of the boundary coupling (the "Warner problem" [4]).

Progress in removing this obstacle was made recently in [5, 6, 8 − 10] (see also [7]). These papers proposed a solution of the Warner problem for B-twisted Landau-Ginzburg models with target space $\mathbb{C}^n$ and for particular families of D-branes described by super-bundles whose rank is constrained to be a power of two. In a slightly modified form, this solution was generalized in [12] by removing unnecessary assumptions, thus giving the general form of the relevant boundary coupling.

As in the closed string case, it is natural to translate the physical data of open Landau-Ginzburg models into the language of singularity theory. For the closed string sector, a crucial step in this regard is the localization formula of [1], which relates topological field theory correlators to residues (see, for example, [13]). An open string version of this formula was proposed in [9], though a complete derivation based on the microscopic boundary coupling was not given.

Given the boundary coupling constructed in [12], we shall re-consider this issue in the more general set-up of [14], and give a complete derivation of this localization formula. Another purpose of this paper is to extend the open and closed localization formulae in a manner which reflects the basic intuition [12] that the B-branes of Landau-Ginzburg models are the result of tachyon condensation between the elementary branes of the B-type sigma model, with tachyon condensation driven by the Landau-Ginzburg superpotential. As we shall show somewhere else, this allows one to make contact with the string field theory approach advocated in a different context in [15 − 25].

Perhaps surprisingly, this is non-trivial to achieve already for the closed string sector. Indeed, the usual on-shell descriptions of the space of bulk observables differ markedly between the two models. In the B-twisted sigma model, this space is described as the $\bar{\partial}$-cohomology of the algebra of $(0, p)$-forms on the target space, valued in holomorphic polyvector fields. In the Landau-Ginzburg case, the space of on-shell bulk observables can be identified with the Jacobi ring of the Landau-Ginzburg superpotential. As we shall see below, these two descriptions are related in a subtle manner, namely by a one-parameter family of "localization pictures" which interpolates between them. The existence of such a family will be established by refining the localization argument of [1]. The different pictures are indexed by a parameter λ, which roughly measures the area of worldsheets with S^2 topology. The B-model description of the algebra of observables arises in the limit when the worldsheet is collapsed to a point, while the Jacobi realization is recovered for very large areas. More precisely, one can construct an off-shell model for each localization picture, with a reduced BRST operator whose cohomology reproduces the space of on-shell bulk observables. The various pictures are related by a "homotopy flow", i.e. a one-parameter semigroup of operators homotopic to the identity. This flow induces the trivial action on BRST cohomology, thus identifying on-shell data between different pictures.

Extending this construction to the boundary sector, we find a similar description. Namely, we shall construct a family of localization pictures indexed by *two* parameters λ and μ, which — when real — measure the area of a worldsheet with disk topology and the length of its boundary. Taking $\mu = 0$, the standard realization of observables in the open B-model arises for $\lambda \to 0$, while the LG description and residue formula of [9] are obtained in the opposite limit $\lambda \to +\infty$. The parameter μ plays an auxiliary role, related to a certain boundary term which was not included in [12] since it is not essential for the topological model. Contrary to previous proposals, we show that this parameter can be safely set to zero, without affecting the localization data. Physically, localization on the critical set of W is controlled by the bulk parameter λ, and is completely independent of the choice of μ. In fact, one must set $\mu = 0$ in order to recover[1] the residue representation of [9].

The paper is organized as follows. In section 2, we review the bulk lagrangian and some of its basic properties, following [14]. In section 3, we discuss localization in the bulk sector. Through careful analysis, we show that one can localize on the zero modes of the *sigma model* action, namely that part of the bulk action which is independent of the Landau-Ginzburg superpotential W. This gives the one-parameter family of localization pictures. We also give the geometric realization of these pictures, and the homotopy flow connecting them. In section 5, we recall the boundary coupling given in [12] and adapt it by adding a supplementary term also suggested in [6, 8, 9] for a special case. This assures that the coupling preserves a full copy of the $N = 2$ topological algebra, when the model is considered on a flat strip. Since the second generator of this algebra is gauged when considering the model on a curved Riemann surface, this condition does not play a fundamental role for unintegrated amplitudes, but the modified coupling is useful for comparison with [9]. Section 5 constructs the boundary observables and correlators, and explains how our model for the boundary BRST operator arises in this approach. In section 6, we discuss localization in the boundary sector. As for the bulk, we proceed by reducing to zero modes of the sigma model action. This gives a family of representations depending on two parameters λ and μ. The second of these weights the contribution arising from the supplementary boundary term. After describing boundary homotopy flows and the associated geometric realization, we construct an off-shell representative for the bulk-boundary map of [28] (see also [26, 27]), and use it to recover (an extension of) the

[1]The authors of [9] propose a different limit, namely $\mu \to +\infty$ with $\lambda = 0$. It does not seem possible to achieve localization on the critical set of W in this limit.

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localization formula proposed in [9], by taking the limit $\lambda \to +\infty$ with $\mu = 0$.[2] Section 7 presents our conclusions. The appendix gives the boundary conditions for the general D-brane coupling.

2. The bulk action

The general formulation of closed B-type topological Landau-Ginzburg models was given in [14], extending the work of [1]. We take the target space to be a Calabi-Yau manifold X, with the Landau-Ginzburg potential a holomorphic function $W \in H^0(\mathcal{O}_X)$. Since any holomorphic function on a compact complex manifold is constant, we shall assume that X is non-compact. In the on-shell formulation, the Grassmann even worldsheet fields are the components ϕ^i, $\phi^{\bar{i}}$ of the map $\phi : \Sigma \to X$, while the G-odd fields are sections η, θ and ρ of the bundles $\phi^*(\bar{T}X)$, $\phi^*(T^*X)$ and $\phi^*(TX) \otimes \mathcal{T}^*\Sigma$ over the worldsheet Σ. Here $\mathcal{T}^*\Sigma$ is the complexified cotangent bundle to Σ, while TX and $\bar{T}X$ are the holomorphic and antiholomorphic components of the complexified tangent bundle $\mathcal{T}X$ to X. This agrees with the on-shell field content of the B-twisted sigma model [29].

2.1 Action and BRST transformations

To write the bulk action, we introduce new fields $\chi, \bar{\chi} \in \Gamma(\Sigma, \phi^*(\bar{T}X))$ by the relations:

$$\eta^{\bar{i}} = \chi^{\bar{i}} + \bar{\chi}^{\bar{i}} \tag{2.1}$$

$$\theta_i = G_{i\bar{j}}(\chi^{\bar{j}} - \bar{\chi}^{\bar{j}})\,. \tag{2.2}$$

We shall also use the quantity $\theta^{\bar{i}} = G^{\bar{i}j}\theta_j$.

As in [14], it is convenient to use an off-shell realization of the BRST symmetry. For this, consider an auxiliary G-even field $\tilde{F}$ transforming as a section of $\phi^*(\mathcal{T}X)$. Then the BRST transformations are:

$$\begin{aligned}
\delta\phi^i &= 0\,, & \delta\phi^{\bar{i}} &= \chi^{\bar{i}} + \bar{\chi}^{\bar{i}} = \eta^{\bar{i}} \\
\delta\chi^{\bar{i}} &= \tilde{F}^{\bar{i}} - \Gamma^{\bar{i}}_{\bar{j}\bar{k}}\bar{\chi}^{\bar{j}}\chi^{\bar{k}}\,, & \delta\bar{\chi}^{\bar{i}} &= -\tilde{F}^{\bar{i}} + \Gamma^{\bar{i}}_{\bar{j}\bar{k}}\bar{\chi}^{\bar{j}}\chi^{\bar{k}} \\
\delta\rho^i_\alpha &= 2\partial_\alpha\phi^i & & \\
\delta\tilde{F}^i &= i\varepsilon^{\alpha\beta}\left[D_\alpha\rho^i_\beta + \frac{1}{4}R^i_{j\bar{l}k}(\chi^{\bar{l}} + \bar{\chi}^{\bar{l}})\rho^j_\alpha\rho^k_\beta\right], & \delta\tilde{F}^{\bar{i}} &= \Gamma^{\bar{i}}_{\bar{j}\bar{k}}\tilde{F}^{\bar{j}}(\chi^{\bar{k}} + \bar{\chi}^{\bar{k}})\,.
\end{aligned} \tag{2.3}$$

These transformations are independent of W. Moreover, the transformations of ϕ, η and ρ do not involve the auxiliary fields. In particular, we have $\delta\eta^{\bar{i}} = 0$. These observations will be used in section 4.

Let us pick a riemannian metric g on the worldsheet. The bulk action of [14] takes the form:

$$S_{\rm bulk} = S_B + S_W \tag{2.4}$$

[2] In this limit, the supplementary term introduced in section 5 does not contribute, so one can use the simplified boundary coupling of [12].

where:

$$S_B = \int_\Sigma d^2\sigma\sqrt{g}\Big[G_{i\bar{j}}\Big(g^{\alpha\beta}\partial_\alpha\phi^i\partial_\beta\phi^{\bar{j}} - i\varepsilon^{\alpha\beta}\partial_\alpha\phi^i\partial_\beta\phi^{\bar{j}} - \frac{1}{2}g^{\alpha\beta}\rho^i_\alpha D_\beta\eta^{\bar{j}} - \\ - \frac{i}{2}\varepsilon^{\alpha\beta}\rho^i_\alpha D_\beta\theta^{\bar{j}} - \tilde{F}^i\tilde{F}^{\bar{j}}\Big) + \frac{i}{4}\varepsilon^{\alpha\beta}R_{i\bar{l}k\bar{j}}\rho^i_\alpha\chi^{\bar{l}}\rho^k_\beta\chi^{\bar{j}}\Big] \tag{2.5}$$

is the action of the B-twisted sigma model and $S_W = S_0 + S_1$ is the potential-dependent term, with:

$$S_0 = -\frac{i}{2}\int_\Sigma d^2\sigma\sqrt{g}\left[D_{\bar{i}}\partial_{\bar{j}}\bar{W}\chi^{\bar{i}}\chi^{\bar{j}} - (\partial_{\bar{i}}\bar{W})\tilde{F}^{\bar{i}}\right] \tag{2.6}$$

$$S_1 = -\frac{i}{2}\int_\Sigma d^2\sigma\sqrt{g}\left[(\partial_i W)\tilde{F}^i + \frac{i}{4}\varepsilon^{\alpha\beta}D_i\partial_j W\rho^i_\alpha\rho^j_\beta\right]. \tag{2.7}$$

The quantity $\varepsilon^{\alpha\beta} = \epsilon^{\alpha\beta}/\sqrt{g}$ is the Levi-Civita tensor, while $\epsilon^{\alpha\beta}$ is the associated density. We have rescaled the Landau-Ginzburg potential W by a factor of $\frac{i}{2}$ with respect to the conventions of [14] (the conventions for the target space Riemann tensor and covariantized worldsheet derivative D_α are unchanged). In S_W, we separated the term depending on W from that depending on its complex conjugate.

As shown in [14], the topological sigma model action (2.5) is BRST exact on closed Riemann surfaces. Since in this paper we shall allow Σ to have a nonempty boundary, we must be careful with total derivative terms. Extending the computation of [14] to this case, we find:

$$S_B + s = \delta V_B \tag{2.8}$$

where:

$$V_B := \int_\Sigma d^2\sigma\sqrt{g}G_{i\bar{j}}\left(\frac{1}{2}g^{\alpha\beta}\rho^i_\alpha\partial_\beta\phi^{\bar{j}} - \frac{i}{2}\varepsilon^{\alpha\beta}\rho^i_\alpha\partial_\beta\phi^{\bar{j}} - \tilde{F}^i\chi^{\bar{j}}\right) \tag{2.9}$$

and:

$$s := i\int_\Sigma d^2\sigma\sqrt{g}\varepsilon^{\alpha\beta}\partial_\alpha(G_{\bar{i}j}\chi^{\bar{i}}\rho^j_\beta) = i\int_\Sigma d(G_{\bar{i}j}\chi^{\bar{i}}\rho^j). \tag{2.10}$$

Since total derivative terms do not change physics on closed Riemann surfaces, we are free to redefine the bulk sigma-model action by adding (2.10) to S_B:

$$\tilde{S}_B := S_B + s = \delta V_B\,. \tag{2.11}$$

Accordingly, we shall use the modified bulk Landau-Ginzburg action:

$$\tilde{S}_{\rm bulk} = S_{\rm bulk} + s = \tilde{S}_B + S_0 + S_1\,. \tag{2.12}$$

It is not hard to check that the term S_0 is BRST exact:

$$S_0 = \delta V_0 \tag{2.13}$$

where:

$$V_0 = \frac{i}{4}\int_\Sigma d^2\sigma\sqrt{g}\theta^{\bar{i}}\partial_{\bar{i}}\bar{W}\,. \tag{2.14}$$

Equations (2.8) and (2.13) are local, i.e. they hold for the associated Lagrange densities without requiring integration by parts. Thus both of these relations can be applied to bordered Riemann surfaces.

Since the boundary term (2.10) is independent of the worldsheet metric, the bulk stress energy tensor has the form given in [14]:

$$\begin{aligned} T_{\mu\nu} &= \frac{1}{2\sqrt{g}}\frac{\delta\tilde{S}_{\rm bulk}}{\delta g^{\mu\nu}} \\ &= \frac{1}{2}G_{i\bar{j}}\left[\partial_\mu\phi^i\partial_\nu\phi^{\bar{j}}+\partial_\nu\phi^i\partial_\mu\phi^{\bar{j}}-\frac{1}{2}\left(\rho^i_\mu D_\nu\eta^{\bar{j}}+\rho^i_\nu D_\mu\eta^{\bar{j}}\right)\right]- \\ &\quad -\frac{1}{2}g_{\mu\nu}\Big[G_{i\bar{j}}g^{\alpha\beta}\left(\partial_\alpha\phi^i\partial_\beta\phi^{\bar{j}}-\frac{1}{2}\rho^i_\alpha D_\beta\eta^{\bar{j}}-\tilde{F}^i\tilde{F}^{\bar{j}}\right)- \\ &\qquad -\frac{i}{2}\partial_i W\tilde{F}^i+\frac{i}{2}\partial_{\bar{i}}\bar{W}\tilde{F}^{\bar{i}}-\frac{i}{4}D_{\bar{i}}\partial_{\bar{j}}\bar{W}\theta^{\bar{i}}\eta^{\bar{j}}\Big]. \end{aligned} \tag{2.15}$$

As explained in [14], $T_{\mu\nu}$ is BRST exact only modulo the equations of motion for the auxiliary fields:

$$\tilde{F}^i=\frac{i}{2}G^{i\bar{j}}\partial_{\bar{j}}\bar{W}\,, \qquad \tilde{F}^{\bar{i}}=-\frac{i}{2}G^{\bar{i}j}\partial_j W\,. \tag{2.16}$$

Imposing these equations, one finds:

$$\begin{aligned} T^{os}_{\mu\nu} &= \frac{1}{2}G_{i\bar{j}}\left[\partial_\mu\phi^i\partial_\nu\phi^{\bar{j}}+\partial_\nu\phi^i\partial_\mu\phi^{\bar{j}}-\frac{1}{2}\left(\rho^i_\mu D_\nu\eta^{\bar{j}}+\rho^i_\nu D_\mu\eta^{\bar{j}}\right)\right]- \\ &\quad -\frac{1}{2}g_{\mu\nu}\left[G_{i\bar{j}}g^{\alpha\beta}\left(\partial_\alpha\phi^i\partial_\beta\phi^{\bar{j}}-\frac{1}{2}\rho^i_\alpha D_\beta\eta^{\bar{j}}\right)+\frac{1}{4}G^{i\bar{j}}\partial_i W\partial_{\bar{j}}\bar{W}-\frac{i}{4}D_{\bar{i}}\partial_{\bar{j}}\bar{W}\theta^{\bar{i}}\eta^{\bar{j}}\right]. \end{aligned} \tag{2.17}$$

This obeys the BRST exactness condition [14]:

$$T^{os}_{\mu\nu}=\delta G_{\mu\nu} \tag{2.18}$$

where:[3]

$$G_{\mu\nu}=\frac{1}{4}\left[G_{i\bar{j}}(\rho^i_\mu\partial_\nu\phi^{\bar{j}}+\rho^i_\nu\partial_\mu\phi^{\bar{j}})-g_{\mu\nu}\left(G_{i\bar{j}}g^{\alpha\beta}\rho^i_\alpha\partial_\beta\phi^{\bar{j}}+\frac{i}{2}\theta^{\bar{i}}\partial_{\bar{i}}\bar{W}\right)\right]. \tag{2.19}$$

On an infinite flat cylinder, the supercharges:

$$G_\mu := \int d\sigma_1 G_{0\mu} \tag{2.20}$$

generate symmetries $\delta_\mu=\{G_\mu,\cdot\}_P$ which together with $\delta=\{Q,\cdot\}_P$ and a supplementary nilpotent transformation $\delta':=\{M,\cdot\}_P$ form the topological algebra of [14] (here $\{\cdot,\cdot\}_P$ is the Poisson bracket of the hamiltonian formulation). When placing the model on a flat strip, the boundary conditions break the symmetries δ' and δ_1, but preserve the subalgebra generated by δ and δ_0. In the untwisted model, this subalgebra corresponds to the usual B-type supersymmetry considered, for example, in [3].

[3]The formula given for $G_{\mu\nu}$ in [14] seems to be missing a global prefactor of $-1/2$.

2.2 BRST variation of the bulk action and the topological Warner term

It is not hard to check that the BRST variation of $\tilde{S}_{\rm bulk}$ produces a boundary term:

$$\delta\tilde{S}_{\rm bulk} = \delta S_1 = \frac{1}{2}\int_{\partial\Sigma} \rho^i \partial_i W \,. \tag{2.21}$$

The presence of a non-zero right hand side in (2.21) is known as the Warner problem [4].

2.3 δ_0-variation of the bulk action on flat Riemann surfaces

When considered on a flat Riemann surface, our model has an enlarged symmetry algebra which was originally described in [14]. In this paper we shall need only the subalgebra obtained by considering an additional odd generator δ_0 beyond the BRST generator of (2.3). In the notations of [14], we have $\delta_0 = \{G_0, \}$, where $G_0 = G_z + G_{\bar{z}}$. Using the results of [14], one finds:

$$\begin{aligned}
\delta_0\phi^i &= \frac{1}{2}\rho_0^i\,, & \delta_0\phi^{\bar{i}} &= 0 \\
\delta_0\rho_0^i &= 0\,, & \delta_0\rho_1^i &= -i\tilde{F}^i \\
\delta_0\eta^{\bar{i}} &= \partial_0\phi^{\bar{i}}\,, & \delta_0\theta^{\bar{i}} &= -i\partial_1\phi^{\bar{i}} \\
\delta_0\tilde{F}^i &= 0\,, & \delta_0\tilde{F}^{\bar{i}} &= \frac{1}{2}(\partial_0\theta^{\bar{i}} + i\partial_1\eta^{\bar{i}})\,.
\end{aligned} \tag{2.22}$$

In this subsection, we are assuming that the worldsheet metric is flat, namely $g_{\alpha\beta} = \delta_{\alpha\beta}$ in the real coordinates σ^0 and σ^1.

If Σ is a cylinder, one finds $\delta_0\tilde{S}_{\rm bulk} = 0$. However, the δ_0 variation of $\tilde{S}_{\rm bulk}$ gives a boundary term when the model is considered on the strip or on the disk. Let us take Σ to be infinite the strip given by $(\sigma^0,\sigma^1) \in \mathbb{R}\times[0,\pi]$. We find:

$$\delta_0\tilde{S}_{\rm bulk} = -\frac{1}{4}\int_{\partial\Sigma} d\tau \eta^{\bar{i}}\partial_{\bar{i}}\bar{W}\,. \tag{2.23}$$

Here $d\tau = d\sigma^0$ is the length element along the boundary of Σ.

3. Localization formula for correlators on the sphere

Let us consider zero-form bulk observables $\mathcal{O}$ which are independent on the auxiliary fields $\tilde{F}^i$ or $\tilde{F}^{\bar{i}}$. We are interested in the sphere correlator of such observables:

$$\langle\mathcal{O}\rangle_{\rm sphere} = \int \mathcal{D}[\phi]\mathcal{D}[\tilde{F}]\mathcal{D}[\eta]\mathcal{D}[\theta]\mathcal{D}[\rho]\; e^{-\tilde{S}_{\rm bulk}}\mathcal{O}\,, \tag{3.1}$$

where we assume that $\mathcal{O}$ is BRST closed.

In this section, we re-consider the localization formula for such correlators. Extending the original argument of [1], we will extract a one-parameter family of representations of (3.1) as a finite-dimensional integral. The basic point is a follows. Since the B-model piece of the Landau-Ginzburg action is BRST exact off-shell, the standard argument of [29] implies that we can localize on the zero modes of the associated *sigma model*. Thus we shall localize on constant maps, without requiring that such maps send the worldsheet to

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the critical points of W. After this reduction, one finds that the lagrangian density of the model becomes BRST exact, so the resulting integral representation is insensitive to multiplying the lagrangian density by a prefactor λ. Since the former appears multiplied by the worldsheet area, this prefactor measures the scale of the underlying S^2 worldsheet. Thus we obtain a one-parameter family of localization formulae for our correlator. Each such "localization picture" allows us to give a geometric representation of genus zero data, thus providing a geometric model for the off-shell state space, BRST operator and bulk trace.

In this approach, the residue representation of [1] is recovered in the limit $\mathrm{Re}\,\lambda \to +\infty$, which forces the point-like image of the worldsheet to lie on the critical set of W. Intuitively, this is the limit of *large* worldsheet areas, the opposite of the "microscopic" limit $\lambda \to 0$. Varying λ allows one to interpolate between these limits, thus connecting the "sigma-model like" and "residue-like" models of genus zero data.

3.1 Localization on B-model zero modes

Since $\tilde{S}_B$ is BRST exact, we can replace the bulk action with:

$$\tilde{S}_{\text{bulk}} = t\tilde{S}_B + S_W = t\delta V_B + S_W\,, \tag{3.2}$$

where t is a complex parameter with $\mathrm{Re}\,t > 0$ (so that the integral is well-defined). BRST invariance of the path integral together with BRST closure of $\mathcal{O}$ imply that the resulting correlator is independent of t. This means that (3.1) can be computed in the limit $\mathrm{Re}\ t \to +\infty$, where the integral localizes on the zero-modes of $\tilde{S}_B$. Since ρ has no zero-modes on the sphere, we must consider configurations for which $\rho^i_\alpha = 0$ while ϕ, η, θ and $\tilde{F}$ are constant on the worldsheet. For such configurations, $\tilde{S}_B$ reduces to:

$$\tilde{S}_B|_{\text{zero modes}} = -AG_{i\bar{j}}\tilde{F}^i\tilde{F}^{\bar{j}}\,, \tag{3.3}$$

with $G_{i\bar{j}} = G_{i\bar{j}}(\phi)$. Here A is the area of the worldsheet. The contribution S_W becomes:

$$S_W|_{\text{zero modes}} = -\frac{i}{2}A\left[D_{\bar{i}}\partial_{\bar{j}}\bar{W}\chi^{\bar{i}}\bar{\chi}^{\bar{j}} - (\partial_{\bar{i}}\bar{W})\tilde{F}^{\bar{i}} + (\partial_i W)\tilde{F}^i\right]. \tag{3.4}$$

Combining these expressions, we find the zero-mode reduction of the worldsheet action:

$$\tilde{S}_0 := \tilde{S}_{\text{bulk}}|_{\text{zero modes}} = -AG_{i\bar{j}}\tilde{F}^i\tilde{F}^{\bar{j}} - \frac{i}{2}A\left[\frac{1}{2}D_{\bar{i}}\partial_{\bar{j}}\bar{W}\theta^{\bar{i}}\eta^{\bar{j}} - (\partial_{\bar{i}}\bar{W})\tilde{F}^{\bar{i}} + (\partial_i W)\tilde{F}^i\right], \tag{3.5}$$

where we wrote χ and $\bar{\chi}$ in terms of η and θ. The correlator (3.1) reduces to an ordinary integral over the zero-modes $\phi^i, \phi^{\bar{i}}$, $\tilde{F}^i, \tilde{F}^{\bar{i}}$ and $\eta^{\bar{i}}, \theta_i$:

$$\langle\mathcal{O}\rangle_{\text{sphere}} = \int d\phi d\tilde{F} d\eta d\theta\, e^{-\tilde{S}_0}\mathcal{O}\,. \tag{3.6}$$

On zero modes, the BRST generator (2.3) takes the form:

$$\begin{aligned}
\delta\phi^i &= 0\,, \qquad & \delta\phi^{\bar{i}} &= \eta^{\bar{i}} \\
\delta\eta^{\bar{i}} &= 0\,, \qquad & \delta\theta^{\bar{i}} &= 2\tilde{F}^{\bar{i}} + \Gamma^{\bar{i}}_{\bar{j}\bar{k}}\theta^{\bar{j}}\eta^{\bar{k}} \\
\delta\tilde{F}^i &= 0\,, \qquad & \delta\tilde{F}^{\bar{i}} &= \Gamma^{\bar{i}}_{\bar{j}\bar{k}}\tilde{F}^{\bar{j}}\eta^{\bar{k}}.
\end{aligned} \tag{3.7}$$

In particular, we have:

$$G_{i\bar{j}}\tilde{F}^i\tilde{F}^{\bar{j}} = \frac{1}{2}\delta[G_{i\bar{j}}\tilde{F}^i\theta^{\bar{j}}]\,, \tag{3.8}$$

which is the zero-mode remnant of equation (2.8). One can also check directly that the reduced action is BRST-closed.

The integral over $\tilde{F}$ can be cast into gaussian form through the change of variables:

$$\tilde{F}^i = \hat{F}^i + \frac{i}{2}G^{i\bar{j}}\partial_{\bar{j}}\bar{W}\,, \qquad \tilde{F}^{\bar{i}} = \hat{F}^{\bar{i}} - \frac{i}{2}G^{\bar{i}j}\partial_j W\,. \tag{3.9}$$

Then the reduced action becomes:

$$\tilde{S}_0 = -AG_{i\bar{j}}\hat{F}^i\hat{F}^{\bar{j}} + \frac{1}{4}A\left[-iD_{\bar{i}}\partial_{\bar{j}}\bar{W}\theta^{\bar{i}}\eta^{\bar{j}} + G^{i\bar{j}}(\partial_i W)(\partial_{\bar{j}}\bar{W})\right]. \tag{3.10}$$

Integrating over $\hat{F}$, we find:

$$\langle\mathcal{O}\rangle_{\text{sphere}} = N\int d\phi d\eta d\theta e^{-\frac{A}{4}\tilde{L}_0}\mathcal{O}\,, \tag{3.11}$$

where:

$$\tilde{L}_0 := -iD_{\bar{i}}\partial_{\bar{j}}\bar{W}\theta^{\bar{i}}\eta^{\bar{j}} + G^{i\bar{j}}(\partial_i W)(\partial_{\bar{j}}\bar{W}) \tag{3.12}$$

plays the role of zero-mode Lagrange density. The prefactor in (3.11) has the form:

$$N = \frac{(2\pi)^n}{A^n\det(G_{i\bar{j}})}\,, \tag{3.13}$$

where n is the complex dimension of the target space X. Since we integrated out the fields $\hat{F}$, the BRST generator on zero-modes reduces to:

$$\begin{aligned} \delta\phi^i &= 0\,, & \delta\phi^{\bar{i}} &= \eta^{\bar{i}} \\ \delta\eta^{\bar{i}} &= 0\,, & \delta\theta^{\bar{i}} &= -iG^{\bar{i}j}\partial_j W + \Gamma^{\bar{i}}_{\bar{j}\bar{k}}\theta^{\bar{j}}\eta^{\bar{k}}\,, \end{aligned} \tag{3.14}$$

which is obtained from (3.7) by imposing the equations of motion:

$$\tilde{F}^i = \frac{i}{2}G^{i\bar{j}}\partial_{\bar{j}}\bar{W}\,, \qquad \tilde{F}^{\bar{i}} = -\frac{i}{2}G^{\bar{i}j}\partial_j W\,. \tag{3.15}$$

For later reference, notice that the last relation in (3.14) is equivalent with:

$$\delta\theta_i = -i\partial_i W\,. \tag{3.16}$$

Using this form of the BRST transformations, one finds that the zero-mode Lagrange density (3.12) is BRST exact:

$$\tilde{L}_0 = \delta\tilde{v}_0\,, \tag{3.17}$$

where:

$$\tilde{v}_0 = i\theta^{\bar{i}}\partial_{\bar{i}}\bar{W}\,. \tag{3.18}$$

Thus we can replace (3.11) by:

$$\langle\mathcal{O}\rangle_{\text{sphere}} = N\int d\phi d\eta d\theta\; e^{-\lambda\tilde{L}_0}\mathcal{O}\,, \tag{3.19}$$

where λ is a complex parameter with positive real part. The integral (3.19) is independent of its value.

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3.2 The space of bulk observables and its cohomology

The observables of interest have the form:

$$\mathcal{O}_\omega(\sigma) := \omega^{j_1\dots j_q}_{\bar{i}_1\dots\bar{i}_p}(\phi(\sigma))\eta^{\bar{i}_1}(\sigma)\dots\eta^{\bar{i}_p}(\sigma)\theta_{j_1}(\sigma)\dots\theta_{j_q}(\sigma)\,, \tag{3.20}$$

where $\omega := \omega^{j_1\dots j_q}_{\bar{i}_1\dots\bar{i}_p}dz^{\bar{i}_1}\wedge\dots\wedge dz^{\bar{i}_p}\wedge\partial_{j_1}\wedge\dots\wedge\partial_{j_q}$ is a section of the bundle $\Lambda^p\bar{T}^*X\wedge\Lambda^qTX$. After reduction to B-model zero modes, we are left with:

$$\mathcal{O}_\omega = \omega^{j_1\dots j_q}_{\bar{i}_1\dots\bar{i}_p}(\phi)\eta^{\bar{i}_1}\dots\eta^{\bar{i}_p}\theta_{j_1}\dots\theta_{j_q}\,, \tag{3.21}$$

which can be identified with the polyvector-valued form ω upon setting:

$$\eta^{\bar{j}} \equiv dz^{\bar{j}}\,, \qquad \theta_j \equiv \partial_j\,. \tag{3.22}$$

The reduced BRST operator (3.14) becomes:

$$\delta \equiv \bar{\partial} + i_{\partial W} \tag{3.23}$$

where i_{∂_W} is the odd derivation of Λ^*TX uniquely determined by the conditions:

$$i_{\partial W}(\partial_j) = -i\partial W(\partial_j) = -i\partial_j W\,. \tag{3.24}$$

Thus the cohomology of the differential superalgebra $(\mathcal{H},\delta)$, where $\mathcal{H} := \Gamma(\Lambda^*\bar{T}^*X\wedge\Lambda^*TX)$, models the algebra of bulk observables. The obvious relations:

$$\bar{\partial}^2 = (i_{\partial W})^2 = \bar{\partial}i_{\partial_W} + i_{\partial_W}\bar{\partial} = 0 \tag{3.25}$$

show that $(\mathcal{H},\delta)$ is a bicomplex. Hence the BRST cohomology is computed by a spectral sequence E_* whose second term equals:

$$E_2 := H_{i_{\partial_W}}(H_{\bar{\partial}}(\mathcal{H}))\,. \tag{3.26}$$

Since the target space is non-compact, we must of course specify a growth condition at infinity. We shall take $\mathcal{H}$ to consist of those sections of the bundle $\Lambda^*\bar{T}^*X\wedge\Lambda^*TX$ whose coefficients have at most polynomial growth. When the spectral sequence collapses to its second term, the BRST cohomology reduces to (3.26). A standard example is the case $X = \mathbb{C}^n$, with W a polynomial function of n variables. Then the $\bar{\partial}$-Poincare Lemma implies that $H_{\bar{\partial}}(\mathcal{H})$ coincides with the space $\Gamma_{poly}(\Lambda^*TX)$ of polyvector fields with polynomial coefficients. In this case, the BRST cohomology reduces to the Jacobi ring $\mathbb{C}[x_1\dots x_n]/\langle\partial_1 W\dots\partial_n W\rangle$, thereby recovering a well-known result.

3.3 The geometric model

Let us translate (3.19) into classical mathematical language. Using (3.22), we find:

$$\tilde{L}_0 \equiv i\bar{H}^i{}_{\bar{j}}dz^{\bar{j}}\wedge\partial_i + G^{i\bar{j}}(\partial_i W)(\partial_{\bar{j}}\bar{W}) \in \mathcal{H}\,. \tag{3.27}$$

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Here $H^i{}_{\bar{j}} := G^{i\bar{k}}\bar{H}_{\bar{k}\bar{j}}$, where $H_{ij} := D_i\partial_j W$ is the Hessian of W. Consider the Hessian operator:

$$H = H^{\bar{i}}{}_j dz^j \otimes \partial_{\bar{i}} \in \mathrm{Hom}(TX, \bar{T}X) = T^*X \otimes \bar{T}X\,, \tag{3.28}$$

whose complex conjugate has the form:

$$\bar{H} = \bar{H}^i{}_{\bar{j}} dz^{\bar{j}} \otimes \partial_i \in \mathrm{Hom}(\bar{T}X, TX) = \bar{T}^*X \otimes TX\,. \tag{3.29}$$

The quantity $\bar{H}_a := \bar{H}^i{}_{\bar{j}} dz^{\bar{j}} \wedge \partial_i \in \bar{T}^*X \wedge TX$ appearing in (3.27) is the antisymmetric part of $\bar{H}$. On the other hand, the second term of (3.27) is the norm of the differential form $\partial W = \partial_i W dz^i$. This gives the coordinate-independent version of (3.27):

$$\tilde{L}_0 = i\bar{H}_a + ||\partial W||^2\,. \tag{3.30}$$

Also note the representation:

$$\tilde{v}_0 = iG^{i\bar{j}}\partial_i \wedge \partial_{\bar{j}}\bar{W}\,. \tag{3.31}$$

It is now easy to see that (3.19) becomes:

$$\mathrm{Tr}\,\omega := \langle \mathcal{O}_\omega \rangle_{\mathrm{sphere}} = N\int_X \Omega \wedge \left[\Omega \lrcorner \left(e^{-\lambda\tilde{L}_0} \wedge \omega\right)\right]\,, \tag{3.32}$$

where $\lrcorner$ denotes the total contraction of a form with a polyvector. The linear functional Tr realizes the bulk trace of [28].

Observation. The integral representation (3.32) allows us to give another (and completely rigorous) proof of λ-independence for $\delta\omega = 0$, with the assumption $\mathrm{Re}\,\lambda > 0$. For this, we have to show that the λ-derivative of (3.32) vanishes. Since $\tilde{L}_0 = \delta\tilde{v}_0$ and $\delta\omega = 0$, this derivative takes the form:

$$\frac{d}{d\lambda}\mathrm{Tr}\,\omega = -\lambda N\int_X \Omega \wedge \left[\Omega \lrcorner \delta\left(e^{-\lambda\tilde{L}_0} \wedge \tilde{v}_0 \wedge \omega\right)\right]\,. \tag{3.33}$$

Thus it suffices to show that $\int_X \Omega \wedge [\Omega \lrcorner \delta\alpha]$ vanishes for any $\alpha \in \mathcal{H}$ which decays exponentially at infinity on X (the exponential decay for $\alpha = e^{-\lambda\tilde{L}_0}\tilde{v}_0 \wedge \omega$ in (3.33) is due to the second term in (3.27)). Notice further that $\int_X \Omega \wedge [\Omega \lrcorner \delta\alpha]$ vanishes for degree reasons unless $\delta\alpha \in \Gamma(\Lambda^n \bar{T}^*X \wedge \Lambda^n TX)$. Hence it is enough to show vanishing of $\int_X \Omega \wedge [\Omega \lrcorner \delta\alpha]$ for an exponentially decaying α such that $\delta\alpha \in \Gamma(\Lambda^n \bar{T}^*X \wedge \Lambda^n TX)$. In this case, we obviously have $\delta\alpha = \bar{\partial}\beta$ for some exponentially decaying $\beta \in \Gamma(\Lambda^{n-1}\bar{T}^*X \wedge \Lambda^n TX)$ (this follows by noticing that the image of $i_{\partial W}$ has vanishing intersection with the subspace $\Gamma(\Lambda^n \bar{T}^*X \wedge \Lambda^n TX)$). Therefore, we only need to show that $\int_X \Omega \wedge [\Omega \lrcorner \bar{\partial}\beta]$ vanishes. This last fact follows from $\Omega \wedge [\Omega \lrcorner \bar{\partial}\beta] = \bar{\partial}(\Omega \wedge [\Omega \lrcorner \beta])$, since the boundary term vanishes due to the exponential decay of β. The assumption $\mathrm{Re}\,\lambda > 0$ is crucial, since otherwise we cannot rely on exponential decay to conclude that the boundary term vanishes.

3.4 Localization pictures and homotopy flows

Expression (3.32) admits the following interpretation. Consider the one-parameter semigroup of operators $U(\lambda)$ acting on $\mathcal{H}$ through wedge multiplication by $e^{-\lambda\tilde{L}_0}$:

$$U(\lambda)\omega := e^{-\lambda\tilde{L}_0}\wedge\omega \quad \text{for all} \ \ \omega\in\mathcal{H}\,. \tag{3.34}$$

The semigroup is defined on the half-plane $\Delta := \{\lambda\in\mathbb{C}|\,\mathrm{Re}\,\lambda>0\}$, so that $U(\lambda)$ maps $\mathcal{H}$ into a subspace of itself. Then (3.32) takes the form:

$$\mathrm{Tr}\,\omega := \mathrm{Tr}^B(U(\lambda)\omega), \tag{3.35}$$

where Tr^B is the bulk trace of the B-twisted sigma model:

$$\mathrm{Tr}^B\,\omega := N\int_X \Omega\wedge(\Omega\lrcorner\omega)\,. \tag{3.36}$$

Since $\tilde{L}_0$ is BRST closed ($\tilde{L}_0=\delta\tilde{v}_0$), each operator $U(\lambda)$ is homotopy equivalent with the identity in the complex $(\mathcal{H},\delta)$:

$$U(\lambda) = 1+[\delta, W_\lambda]\,, \tag{3.37}$$

for some operator W_λ. In particular, $U(\lambda)$ is an endomorphism of our complex, i.e. the following relation holds:

$$U(\lambda)\circ\delta = \delta\circ U(\lambda)\,. \tag{3.38}$$

Such a semigroup will be called a *homotopy flow*. It is clear that each $U(\lambda)$ is a quasi-isomorphism, i.e. induces an automorphism $U_*(\lambda)$ on the BRST cohomology $H_\delta(\mathcal{H})$. Following relation (3.35), we define the *localization picture* λ by associating $\omega_\lambda := U(\lambda)(\omega)\in\mathcal{H}$ to each $\omega\in\mathcal{H}$ (then ω_λ is the representative of the "state" ω in the picture λ). The representatives of this picture belong to the subspace $\mathcal{H}_\lambda := U(\lambda)(\mathcal{H})\subset\mathcal{H}$. As in quantum mechanics, we have a representative for any operator $T\in\mathrm{End}(\mathcal{H})$ in the localization picture λ:

$$T_\lambda := U(\lambda)\circ T\circ U(-\lambda)\in\mathrm{End}(\mathcal{H}_\lambda)\,, \tag{3.39}$$

where $\mathrm{Re}\,\lambda>0$ and $U(-\lambda)$ is defined as an operator from $\mathcal{H}_\lambda$ to $\mathcal{H}$. Relation (3.38) shows that the BRST operator is "picture-independent" in the following sense:

$$Q_\lambda = Q|_{\mathcal{H}_\lambda}\,, \tag{3.40}$$

where in the right hand side we restrict both the domain and image to $\mathcal{H}_\lambda$. Relation (3.35) becomes:

$$\mathrm{Tr}\,\omega = \mathrm{Tr}^B\,\omega_\lambda\,. \tag{3.41}$$

For $\lambda=W=0$, we have $U(0)=Id_{\mathcal{H}}$ and we recover the familiar data of the B-twisted sigma model. Namely, $\mathcal{H}$ provides a geometric model for the off-shell state space, the Dolbeault operator $\bar\partial$ models the "localized BRST operator" and Tr^B models the bulk trace of [28]. Turning on the Landau-Ginzburg superpotential W and performing localization as above with "worldsheet area" λ leads to a geometric model given by the triplet $(\mathcal{H},\delta,\mathrm{Tr})$.

This is related to the triplet describing the B-twisted sigma model by the modification $\delta = \bar{\partial} + i_{\partial W}$ of the BRST operator, followed by the homotopy flow $U(\lambda)$.

Because varying λ along the real axis amounts to changing the area of the worldsheet, the operators $U(\lambda)$ implement a sort of "renormalization group flow" connecting the point-like (UV) limit $\lambda = 0$ with the large area (IR limit) $\lambda = +\infty$. Since the model is topological, such a flow "does nothing" at the level of BRST cohomology, but acts non-trivially off-shell.

3.5 The residue formula for sphere correlators

Since the integral (3.19) is independent of λ, we can compute its value for $\mathrm{Re}\,\lambda \to +\infty$ (more specifically, we shall take $\lambda \to +\infty$ with $\lambda \in \mathbb{R}$). In this limit, the second term in (3.12) forces the integral to localize on the critical points of W, and the gaussian approximation around these points becomes exact. For simplicity, we shall assume that the critical points of W are isolated (the general case can be incorporated by a continuity argument). For simplicity, we shall also assume that the spectral sequence of Subsection 3.2 collapses to its second term.

Taking $\lambda \to +\infty$ with $\lambda \in \mathbb{R}$, we find that the correlator (3.19) vanishes unless $\omega = f$ with f a complex-valued function defined on X.[4] In this case, we obtain:

$$\begin{aligned}\langle \mathcal{O}_f \rangle_{\text{sphere}} &= \lim_{\lambda \to +\infty} \left(N \sum_{p \in \mathrm{Crit}W} \left[n!^2 (i\lambda)^n (-1)^{n(n-1)/2} \det(\bar{H}_{\bar{i}\bar{j}}) \right] \times \right. \\ &\qquad \left. \times \left[\frac{(2\pi)^n}{\lambda^n \det(H_{ij}) \det(\bar{H}_{\bar{i}\bar{j}})} \det(G_{i\bar{j}}) \right] f|_p + O\left(\frac{1}{\lambda}\right) \right) \\ &= n!^2 (2\pi)^n (-1)^{n(n+1)/2} N \det(G_{i\bar{j}}) \sum_{p \in \mathrm{Crit}W} \frac{1}{\det(H_{ij}(p))} f(p)\,. \qquad (3.42)\end{aligned}$$

Since $\sum_{p\in \mathrm{Crit}W} \frac{1}{\det(H_{ij}(p))} f(p) \propto \int_X \frac{f(z) dz^1 \wedge \dots \wedge dz^n}{\partial_1 W \dots \partial_n W}$ by residue theory [13], one recovers the following generalization of the well-known result of [1]:

$$\langle \mathcal{O}_\omega \rangle_{\text{sphere}} = 0 \qquad \text{unless } \omega = f \qquad (3.43)$$

$$\langle \mathcal{O}_f \rangle_{\text{sphere}} = C \int_X \Omega \frac{f(z)}{\partial_1 W \dots \partial_n W}\,. \qquad (3.44)$$

Here C is an uninteresting normalization constant.

4. The boundary coupling

In this section we discuss the boundary coupling of our models. The construction is based on [12], with a certain modification which will prove useful later on. After recalling the

[4] For this, notice that the bosonic gaussian integral over fluctuations of ϕ around each critical point of W produces a factor which is weighted by $\frac{1}{\lambda^n}$. Thus the fermionic gaussian integral over θ and η must produce n powers of λ if one is to obtain a non-vanishing result in the limit $\lambda \to \infty$. This obviously requires that $\mathcal{O}_\omega$ contain no η's or θ's, so that the highest (n-th order term) in the expansion of $e^{+i\lambda D_{\bar{i}} \partial_{\bar{j}} \bar{W} \theta^{\bar{i}} \eta^{\bar{j}}}$ survives when performing the integral over η and θ.

basics of superconnections, we construct the coupling in the form of [12], with the addition of a term which insures δ_0-invariance on a flat strip. While this does not affect the target space equations of motion, it will help us make contact with previous work on the subject. We also give the target space reflection of the δ_0-invariance constraint.

4.1 Mathematical preparations

Consider a complex superbundle $E = E_+ \oplus E_-$ over X, and a superconnection [31] $\mathcal{B}$ on E. We let $r_\pm := \mathrm{rk} E_\pm$. The bundle of endomorphisms $\mathrm{End}(E)$ is endowed with the natural $\mathbb{Z}_2$ grading, with even and odd components:

$$\mathrm{End}_+(E) := \mathrm{End}(E_+) \oplus \mathrm{End}(E_-) \tag{4.1}$$

$$\mathrm{End}_-(E) := \mathrm{Hom}(E_+, E_-) \oplus \mathrm{Hom}(E_-, E_+) \,. \tag{4.2}$$

The superconnection $\mathcal{B}$ can be viewed as a section of $[\mathcal{T}^* X \otimes \mathrm{End}_+(E)] \oplus \mathrm{End}_-(E)$. In a local frame of E compatible with the grading, this is a matrix:

$$\mathcal{B} = \begin{bmatrix} A^{(+)} & F \\ G & A^{(-)} \end{bmatrix} \tag{4.3}$$

whose diagonal entries $A^{(\pm)}$ are connection one-forms on $E_\pm$, while F, G are elements of $\mathrm{Hom}(E_-, E_+)$ and $\mathrm{Hom}(E_+, E_-)$. We require that the superconnection has type $(0, \leq 1)$, i.e. the one-forms $A^{(\pm)}$ belong to $\Omega^{(0,1)}(\mathrm{End}(E_\pm))$. The morphism F should not be confused with the curvature form used below.

When endowed with the ordinary composition of morphisms, the space of sections $\Gamma(\mathrm{End}(E))$ becomes an associative superalgebra. The space $\mathcal{H}_b := \Omega^{(0,*)}(\mathrm{End}(E))$ also carries an associative superalgebra structure, which is induced from $(\Omega^{(0,*)}(X), \wedge)$ and $(\Gamma(\mathrm{End}(E)), \circ)$ via the tensor product decomposition:

$$\Omega^{(0,*)}(\mathrm{End}(E)) = \Omega^{(0,*)}(X) \otimes_{\Omega^{(0,0)}(X)} \Gamma(\mathrm{End}(E)) \,. \tag{4.4}$$

For decomposable elements $u = \omega \otimes f$ and $v = \eta \otimes g$, with homogeneous ω, η and f, g, the associative product on $\mathcal{H}_b$ takes the form:

$$uv = (-1)^{\deg f \ \mathrm{rk}\eta} (\omega \wedge \eta) \otimes (f \circ g) \,, \tag{4.5}$$

where deg denotes the grading of the superalgebra $\mathrm{End}(E)$:

$$\deg(f) = 0 \in \mathbb{Z}_2 \quad \text{if } f \in \mathrm{End}_+(E) \,, \qquad \deg(f) = 1 \in \mathbb{Z}_2 \quad \text{if } f \in \mathrm{End}_-(E) \,. \tag{4.6}$$

The total degree on $\mathcal{H}_b$ is given by:

$$|\omega \otimes f| = \mathrm{rk}\omega + \deg f \pmod 2 \,. \tag{4.7}$$

We also recall the supertrace on $\mathrm{End}(E)$:

$$\mathrm{str}(f) = \mathrm{tr}\, f_{++} - \mathrm{tr}\, f_{--} \,, \tag{4.8}$$

where $f = \left[\begin{smallmatrix} f_{++} & f_{-+} \\ f_{+-} & f_{--} \end{smallmatrix}\right]$ is an endomorphism of E with components $f_{\alpha\beta} \in \mathrm{Hom}(E_\alpha, E_\beta)$ for $\alpha, \beta = +, -$. This has the property:

$$\mathrm{str}(f \circ g) = (-1)^{\deg f \deg g} \mathrm{str}(g \circ f) \tag{4.9}$$

for homogeneous elements f, g.

The twisted Dolbeault operator:

$$\bar{\mathcal{D}} = \bar{\partial} + \mathcal{B} = \begin{bmatrix} \bar{\partial} + A^{(+)} & F \\ G & \bar{\partial} + A^{(-)} \end{bmatrix} \tag{4.10}$$

induces an odd derivation $\bar{\partial} + [\mathcal{B}, \cdot]$ of the superalgebra $\mathcal{H}_b$, where $[u, v] := uv - (-1)^{|u||v|} vu$ is the supercommutator.

The $(0, \leq 2)$ part of the superconnection's curvature has the form:

$$\mathcal{F}^{(0,\leq 2)} = \bar{\mathcal{D}}^2 = \bar{\partial}\mathcal{B} + \frac{1}{2}[\mathcal{B}, \mathcal{B}] = \bar{\partial}\mathcal{B} + \mathcal{B}\mathcal{B} = \begin{bmatrix} F^{(+)}_{(0,2)} + FG & \bar{\nabla} F \\ \bar{\nabla} G & F^{(-)}_{(0,2)} + GF \end{bmatrix} \tag{4.11}$$

where $F^{(\pm)}_{(0,2)}$ are the $(0, 2)$ pieces of the curvature forms $F^{(\pm)}$ of $A^{(\pm)}$ and:

$$\begin{aligned} \bar{\nabla} F &= \bar{\partial} F + A^{(+)} F + F A^{(-)} = \bar{\partial} F + A^{(+)} \circ F - F \circ A^{(-)} \\ \bar{\nabla} G &= \bar{\partial} G + A^{(-)} G + G A^{(+)} = \bar{\partial} G + A^{(-)} \circ G - G \circ A^{(+)} \,. \end{aligned} \tag{4.12}$$

We will use the the notations:

$$A := A^{(+)} \oplus A^{(-)} = \begin{bmatrix} A^{(+)} & 0 \\ 0 & A^{(-)} \end{bmatrix}, \qquad D := \begin{bmatrix} 0 & F \\ G & 0 \end{bmatrix} \tag{4.13}$$

for the diagonal and off-diagonal parts of $\mathcal{B}$. Then A is an connection one-form on E (compatible with the grading), while D is an odd endomorphism. We have $\mathcal{B} = A + D$ and:

$$\mathcal{F}^{(0,\leq 2)} = F^{(0,2)} + \bar{\nabla}_A D + D^2 \,. \tag{4.14}$$

Here $F^{(0,2)} = F^{(+)}_{(0,2)} + F^{(-)}_{(0,2)}$ is the $(0, 2)$ part of the curvature of A and $\bar{\nabla}_A = \bar{\partial} + [A, \cdot]$ is the Dolbeault operator twisted by A.

4.2 The boundary coupling

Following [12], we define the partition function on a bordered and oriented Riemann surface Σ by:

$$Z := \int \mathcal{D}[\phi]\mathcal{D}[\tilde{F}]\mathcal{D}[\theta]\mathcal{D}[\rho]\mathcal{D}[\eta] e^{-\tilde{S}_{\mathrm{bulk}}} \mathcal{U}_1 \ldots \mathcal{U}_h \,, \tag{4.15}$$

where h is the number of holes and the factors $\mathcal{U}_a$ have the form:

$$\mathcal{U}_a := \mathrm{Str}\, P e^{-\oint_{C_a} d\tau_a M} \,. \tag{4.16}$$

We are assuming that the boundary of Σ is a disjoint union of smooth circles C_a, associated with holes labeled by a. The symbol Str denotes the supertrace on $GL(r_+|r_-)$, while $d\tau_a$

stands for the length element along C_a induced by the metric on the interior of Σ. The quantity M is given by:

$$M = \begin{bmatrix} \hat{\mathcal{A}}^{(+)} + \frac{i}{2}(FF^\dagger + G^\dagger G) & \frac{1}{2}\rho_0^i \nabla_i F + \frac{i}{2}\eta^{\bar{i}}\nabla_{\bar{i}} G^\dagger \\ \frac{1}{2}\rho_0^i \nabla_i G + \frac{i}{2}\eta^{\bar{i}}\nabla_{\bar{i}} F^\dagger & \hat{\mathcal{A}}^{(-)} + \frac{i}{2}(F^\dagger F + GG^\dagger) \end{bmatrix} . \tag{4.17}$$

Here $\rho_0^i d\tau_a$ is the pull-back of ρ^i to C_a and:

$$\hat{\mathcal{A}}^{(\pm)} := A_{\bar{i}}^{(\pm)} \dot{\phi}^{\bar{i}} + \frac{1}{2}\eta^{\bar{i}} F_{\bar{i}j}^{(\pm)} \rho_0^j \tag{4.18}$$

are connections on the bundles $\mathcal{E}_\pm$ obtained by pulling back $E_\pm$ to the boundary of Σ. The dot in (4.18) stands for the derivative $\frac{d}{d\tau_a}$. Notice that $\nabla_i F = \partial_i F$ and $\nabla_i G = \partial_i G$ since A is a $(0,1)$-connection.

We have:

$$M = \hat{\mathcal{A}} + \Delta + K \tag{4.19}$$

where:

$$\Delta := \frac{1}{2}\rho_0^i \partial_i D \, , \tag{4.20}$$

$$K := \frac{i}{2}\left(\eta^{\bar{i}}\nabla_{\bar{i}} D^\dagger + [D, D^\dagger]_+\right) \tag{4.21}$$

and:

$$\hat{\mathcal{A}} = \dot{\phi}^{\bar{i}} A_{\bar{i}} + \frac{1}{2} F_{\bar{i}j}\eta^{\bar{i}}\rho_0^j \, . \tag{4.22}$$

Here A is the direct sum connection on $\mathrm{End}(E)$ introduced in (4.13). The first two terms in (4.19) agree with [12], while the last term K is added for comparison with [9]. As we shall see below, this term preserves BRST-invariance of the partition function (which is already preserved by the sum of the first two terms [12]). As for the open B-model, adding K insures invariance of the boundary coupling with respect to the second generator δ_0 of the $N = 2$ topological algebra, thereby fixing an ambiguity familiar form Hodge theory.[5] This modification has minor effects which can be safely ignored for most purposes.[6]

4.3 The target space equations of motion

To insure BRST invariance of the partition function (4.15), we must choose the background superconnection $\mathcal{B}$ such that:

$$\delta \mathcal{U}_a = \frac{1}{2}\left[\int_{C_a} d\tau \rho_0^i \partial_i W\right] \mathcal{U}_a \, . \tag{4.23}$$

[5]The symmetry generators δ and δ_0 can be viewed as analogues of the operators $\bar{\partial}$ and $\bar{\partial}^\dagger$ of Hodge theory, as already pointed out in [29] in the context of twisted B-models. The boundary coupling of [12] is chosen to preserve BRST invariance of the partition function. This is ambiguous up to addition of 'exact' terms, an ambiguity which we can fix by requiring δ_0-invariance of the partition function.

[6]As we shall see in the next section, the extra-term in the boundary coupling can be used to introduce a parameter μ characterizing boundary localization pictures. For most practical purposes, this parameter can be set to zero, which amounts to neglecting the last term in (4.19). In particular, one must set μ to zero in order to recover the trace formula of [9]. It is the *bulk* parameter λ which must be taken to infinity in order to recover the proposal of [9].

In this paper, we also require δ_0-invariance of the partition function on the flat strip:

$$\delta_0 \mathcal{U}_a = -\frac{1}{4}\left[\int_{C_a} d\tau \eta^{\bar{i}} \partial_i \bar{W}\right] \mathcal{U}_a \,. \tag{4.24}$$

It is not hard to check the relations:

$$\delta \mathcal{U}_a = -\,\mathrm{Str}\left[\; I_a(\delta M) P e^{-\oint_{C_a} d\tau_a M}\right] \tag{4.25}$$

where:

$$I_a(\delta M) = \oint_{C_a} d\tau_a U_a^{-1}\Big(F_{\bar{i}\bar{j}} \eta^{\bar{i}} \dot{\phi}^{\bar{j}} - \frac{1}{4} \nabla_k F_{\bar{i}\bar{j}} \eta^{\bar{i}} \eta^{\bar{j}} \rho_0^k - \dot{\phi}^{\bar{i}} \nabla_{\bar{i}} D - \frac{1}{2} \rho_0^i \nabla_i (D^2) + \\ + \frac{1}{2} \eta^{\bar{i}} \rho_0^j \nabla_j \nabla_{\bar{i}} D + + \frac{1}{2} \eta^{\bar{i}} \eta^{\bar{j}} [F_{\bar{i}\bar{j}}, D^\dagger] + \eta^{\bar{i}} [\nabla_{\bar{i}} D, D^\dagger] + [D^2, D^\dagger] \Big) U_a \tag{4.26}$$

and:

$$\delta_0 \mathcal{U}_a = -\,\mathrm{Str}\left[\; I_a(\delta_0 M) P e^{-\oint_{C_a} d\tau_a M}\right] \tag{4.27}$$

where:

$$I_a(\delta_0 M) = \frac{1}{4} \oint_{C_a} d\tau_a U_a^{-1} \left(\eta^{\bar{i}} \nabla_{\bar{i}} (D^\dagger)^2 + [D, (D^\dagger)^2] \right) U_a + \\ + \frac{i}{2} \oint_{C_a} d\tau_a U_a^{-1} \left(-\dot{\phi}^i \nabla_i D^\dagger + \frac{1}{2} \eta^{\bar{i}} \nabla_{\bar{i}} \nabla_j D^\dagger \rho_0^j + \frac{1}{2} \rho_0^i [D, \nabla_i D^\dagger] \right) U_a \,.$$

Here $U_a(\tau_a) \in GL(r_+|r_-)$ is a certain invertible operator[7]) which plays the role of 'parallel transport' defined by M along C_a (see [12] for details). Namely $U_a(\tau_a) = U_a(\tau_a, 0)$, where:

$$U_a(\tau_2, \tau_1) := P e^{-\int_{\tau_1}^{\tau_2} M(\tau) d\tau} \tag{4.28}$$

if $\tau_2 > \tau_1$. The origin of the proper length coordinate τ_a along C_a is chosen arbitrarily, while the orientation on C_a is compatible with that of Σ. The quantities $F_{\bar{i}\bar{j}}$ etc. are the $(0,2)$-components of the curvature of the direct sum connection A introduced in (4.13). Notice the relations:

$$\mathcal{U}_a = \mathrm{Str}\, H_a(\tau) \tag{4.29}$$

where:

$$H_a(\tau) = U(\tau + l_a, \tau) \tag{4.30}$$

are the "superholonomy operators" (here l_a the length of C_a).

Relations (4.23), (4.24) and (4.25), (4.27) show that the BRST and δ_0-invariance conditions amount to:

$$F_{\bar{i}\bar{j}} = 0 \tag{4.31}$$

$$\nabla_{\bar{i}} D = 0 \tag{4.32}$$

$$\nabla_i (D^2) = \partial_i W \tag{4.33}$$

[7]This should not be confused with the homotopy flow of Subsection 3.4!

$$[D^\dagger, D^2] = 0\,. \tag{4.34}$$

The first relation says that A is integrable, so it defines a complex structure on the bundle E. The second condition means that $D \in \mathrm{End}(E)$ is holomorphic with respect to this complex structure. The third equation requires $D^2 = c + W\mathrm{id}_E$, with c a covariantly-constant endomorphism. Comparing with (4.14), we find that these first three conditions are equivalent with:

$$\mathcal{F}^{(0,\leq 2)} = c + W\mathrm{id}_E \Longleftrightarrow \bar{\mathcal{D}}^2 = c + W\mathrm{id}_E\,. \tag{4.35}$$

This is the target space equation of motion for our open string background [12]. Notice that (4.35) admit solutions only when $r_+ = r_-$.

For backgrounds satisfying the equation of motion, the last condition in (4.31) reads:

$$[D^\dagger, D^2] = 0 \Longleftrightarrow [D^\dagger, c] = 0\,. \tag{4.36}$$

This can be viewed as a partial "gauge-fixing" constraint, which is fulfilled, for example, if one takes c to be proportional to the identity endomorphism (in which case the proportionality constant can be absorbed into W). For simplicity, we shall take $c = 0$ for the remainder of this paper.

5. Boundary observables and correlators

As we shall see in section 6, the boundary conditions derived from the partition function (4.15) constrain θ in terms of η along the boundary of the worldsheet. Hence it suffices to consider boundary observables of the form:

$$\mathcal{O}_\alpha(\tau) = \alpha_{\bar{i}_1\ldots\bar{i}_p}(\phi(\tau))\eta^{\bar{i}_1}(\tau)\ldots\eta^{\bar{i}_p}(\tau)\,, \tag{5.1}$$

where τ is a point on $\partial\Sigma$. Here $\alpha := \alpha_{\bar{i}_1\ldots\bar{i}_p}dz^{\bar{i}_1}\wedge\cdots\wedge dz^{\bar{i}_p}$ is a $(0,p)$ form valued in $\mathrm{End}(E)$.

Consider a collection of m topological D-branes described by superbundles E_a endowed with superconnections $\mathcal{B}_a$, such that the target space equations of motion are satisfied. The index a runs from 1 to m. Let Σ be a Riemann surface with m circle boundary components C_a, which we endow with the orientation induced from Σ. Choosing forms $\alpha_j^{(a)} \in \Omega^{(0,p_j^{(a)})}(\mathrm{End}(E_a))$ and points $\tau_1^{(a)}\ldots\tau_{k_a}^{(a)}$ arranged in increasing cyclic order along C_a, we are interested in the correlator:

$$\begin{aligned}\left\langle \prod_{a=1}^{m}\prod_{j_a=k_a}^{1}\mathcal{O}_{\alpha_{j_a}^{(a)}}(\tau_{j_a}^{(a)})\right\rangle_\Sigma := \int \mathcal{D}[\phi]\mathcal{D}[F]\mathcal{D}[\rho]\mathcal{D}[\eta]\mathcal{D}[\theta]e^{-\tilde{S}_{\mathrm{bulk}}}\times \\ \times\prod_{a=1}^{m}\mathrm{Str}\Big[\mathcal{O}_{\alpha_{k_a}^{(a)}}(\tau_{k_a}^{(a)})U_a(\tau_{k_a}^{(a)},\tau_{k_a-1}^{(a)})\mathcal{O}_{\alpha_{k_a-1}^{(a)}}\times \\ (\tau_{k_a-1}^{(a)})\ldots\mathcal{O}_{\alpha_1^{(a)}}(\tau_1^{(a)})U_a(\tau_1^{(a)},\tau_{k_a}^{(a)})\Big]\,,\end{aligned} \tag{5.2}$$

where we used the "parallel supertransport" operators defined in (4.28). The integration domain in (5.2) is specified by the appropriate boundary conditions on the worldsheet fields, which will be discussed in more detail below.

Let us first consider a single operator insertion $\mathcal{O}_\alpha$ along a circle boundary component C. In this case, the relevant factor in (5.2) is:

$$\mathrm{Str}[H(\tau)\mathcal{O}_\alpha(\tau)]\,. \tag{5.3}$$

We wish to compute the BRST variation of this quantity. From the relation:

$$\delta H(\tau) = \left[\frac{1}{2}\int_C \rho^i \partial_i W\right] H(\tau) + [H(\tau), D(\tau) + A_{\bar{i}}(\tau)\eta^{\bar{i}}(\tau)] \tag{5.4}$$

we obtain:

$$\delta\,\mathrm{Str}[H(\tau)\mathcal{O}_\alpha(\tau)] = \left[\frac{1}{2}\int_C \rho^i \partial_i W\right] \mathrm{Str}[H(\tau)\mathcal{O}_\alpha(\tau)] + \mathrm{Str}[H(\tau)\delta_b\mathcal{O}_\alpha(\tau)]\,, \tag{5.5}$$

where:

$$\delta_b\mathcal{O}_\alpha := \delta\mathcal{O}_\alpha + [D + A_{\bar{i}}\eta^{\bar{i}}, \mathcal{O}_\alpha]\,. \tag{5.6}$$

Using the target space equations of motion, one easily checks that[8] δ_b squares to zero, so that it plays the role of an 'effective' BRST operator in the boundary sector. Notice that δ_b arises naturally due to the second term in the BRST variation (5.4) of $H(\tau)$. Using (5.6), we find the relation:

$$\delta_b\mathcal{O}_\alpha = \mathcal{O}_{\bar{D}\alpha} \tag{5.8}$$

where $\bar{D} = \bar{D}_{\mathcal{B}}$ is the Dolbeault operator on $\Omega^{(0,*)}(\mathrm{End}(E))$ twisted by the superconnection $\mathcal{B}$.

It is not hard to generalize (5.5) to the case of k insertions along C:

$$\begin{aligned}
&\delta\,\mathrm{Str}[\mathcal{O}_{\alpha_k}(\tau_k)U(\tau_k,\tau_{k-1})\dots\mathcal{O}_{\alpha_1}(\tau_1)U(\tau_1,\tau_k)] = \\
&\quad = \left[\frac{1}{2}\int_C \rho^i\partial_i W\right]\mathrm{Str}[\mathcal{O}_{\alpha_k}(\tau_k)U(\tau_k,\tau_{k-1})\dots\mathcal{O}_{\alpha_1}(\tau_1)U(\tau_1,\tau_k)] + \\
&\qquad + \sum_{j=1}^k \mathrm{Str}[\mathcal{O}_{\alpha_k}(\tau_k)\dots U(\tau_{j+1},\tau_j)\delta_b\mathcal{O}_{\alpha_j}(\tau_j)U(\tau_j,\tau_{j-1})\dots\mathcal{O}_{\alpha_1}(\tau_1)U(\tau_1,\tau_k)]\,.
\end{aligned} \tag{5.9}$$

Applying this to (5.2), we find that the BRST variation of $e^{-\tilde{S}_{\mathrm{bulk}}}$ is canceled by the first contribution in (5.9), summed over circle boundary components. This gives:

$$\begin{aligned}
&\delta\left(e^{-\tilde{S}_{\mathrm{bulk}}}\prod_{a=1}^m \mathrm{Str}\left[\mathcal{O}_{\alpha^{(a)}_{k_a}}(\tau^{(a)}_{k_a})U(\tau^{(a)}_{k_a},\tau^{(a)}_{k_a-1})\dots\mathcal{O}_{\alpha^{(a)}_1}(\tau^{(a)}_1)U(\tau^{(a)}_1,\tau^{(a)}_{k_a})\right]\right) = \\
&\quad = e^{-\tilde{S}_{\mathrm{bulk}}}\sum_{a=1}^m\sum_{j_a=1}^{k_a}\mathrm{Str}\left[\mathcal{O}_{\alpha^{(a)}_{k_a}}(\tau^{(a)}_{k_a})U(\tau^{(a)}_{k_a},\tau^{(a)}_{k_a-1})\dots\delta_b\mathcal{O}_{\alpha^{(a)}_{j_a}}(\tau^{(a)}_{j_a})\dots\mathcal{O}_{\alpha^{(a)}_1}(\tau^{(a)}_1)\times\right. \\
&\qquad\qquad \left. U(\tau^{(a)}_1,\tau^{(a)}_{k_a})\right]\,.
\end{aligned} \tag{5.10}$$

[8]Indeed, one has:

$$\delta_b^2\mathcal{O} = \frac{1}{2}[F_{\bar{i}\bar{j}},\mathcal{O}] + \eta^{\bar{i}}[\nabla_{\bar{i}}D,\mathcal{O}] + [D^2,\mathcal{O}]\,. \tag{5.7}$$

Equation (5.10) replaces the more familiar formula known from the open topological sigma model. Unlike the sigma model case, the left hand side includes the factor $e^{-\tilde{S}_{\rm bulk}}$, because its BRST variation does not vanish separately. Equation (5.10) implies that the correlator of δ_b-closed boundary observables only depends on their δ_b-cohomology class, and in particular such a correlator vanishes if one of the boundary observables is δ_b-exact. Remember that $\tilde{S}_{\rm bulk} = \tilde{S}_B + S_W$. BRST closure of $\tilde{S}_B$ implies that (5.10) is equivalent with:

$$\delta\left(e^{-S_W}\prod_{a=1}^{m}\mathrm{Str}\left[\mathcal{O}_{\alpha^{(a)}_{k_a}}(\tau^{(a)}_{k_a})U(\tau^{(a)}_{k_a},\tau^{(a)}_{k_a-1})\dots\mathcal{O}_{\alpha^{(a)}_1}(\tau^{(a)}_1)U(\tau^{(a)}_1,\tau^{(a)}_{k_a})\right]\right) = \tag{5.11}$$

$$= e^{-S_W}\sum_{a=1}^{m}\sum_{j_a=1}^{k_a}\mathrm{Str}\left[\mathcal{O}_{\alpha^{(a)}_{k_a}}(\tau^{(a)}_{k_a})U(\tau^{(a)}_{k_a},\tau^{(a)}_{k_a-1})\dots\delta_b\mathcal{O}_{\alpha^{(a)}_{j_a}}(\tau^{(a)}_{j_a})\dots\mathcal{O}_{\alpha^{(a)}_1}(\tau^{(a)}_1)\times\right.$$
$$\left.\times\, U(\tau^{(a)}_1,\tau^{(a)}_{k_a})\right], \tag{5.12}$$

a fact which will be used in section 6.

Observation. It is easy to extend the discussion above by including boundary condition changing observables, which in the present context have the form (5.1), but with α an element of $\Omega^{(0,p)}(\mathrm{Hom}(E_a,E_b))$. In this case, the operator (5.6) is replaced by:

$$\delta_b\mathcal{O}_\alpha := \delta\mathcal{O}_\alpha + (D^{(b)} + A^{(b)}_{\bar{i}}\eta^{\bar{i}})\mathcal{O}_\alpha - (-1)^{\mathrm{rk}\alpha}\mathcal{O}_\alpha(D^{(a)} + A^{(a)}_{\bar{i}}\eta^{\bar{i}}) \tag{5.13}$$

and $\bar{D}$ in relation (5.8) becomes the Dolbeault operator on $\Omega^{(0,*)}(\mathrm{Hom}(E_a,E_b))$, twisted by the superconnections $\mathcal{B}_a$ and $\mathcal{B}_b$.

6. Localization formula for boundary correlators on the disk

We next discuss localization in the boundary sector. As for the bulk, we will proceed by localizing on sigma model zero-modes, thereby extracting a two-parameter family of localization formulae. The first index of this family is the bulk parameter λ of section 3, while second parameter μ is associated with the last term in (4.19). These two parameters measure the area and circumference length of a worldsheet with disk topology. Each pair (λ,μ) defines a localization picture, and a certain off-shell representation of the boundary trace of [26 − 28]. As we shall see below, the various pictures are again related by a homotopy flow, and in particular the various representations of the boundary trace agree when reduced to the cohomology of δ_b. In this approach, the appropriate generalization of the residue representation of [9] is recovered in the limit $\lambda \to +\infty$ with $\mu = 0$.

The boundary conditions induced by the coupling (4.15) can be extracted by studying the Euler-Lagrange variations of the non-local action $S_{\rm eff} = \tilde{S}_{\rm bulk} - \ln\mathcal{U}$. These conditions are given explicitly in appendix A, where we also show that they are BRST invariant modulo the equations of motion for the auxiliary fields F. The disk correlator of a collection of boundary observables $\mathcal{O}_{\alpha_1}(\tau_1)\dots\mathcal{O}_{\alpha_k}(\tau_k)$ (with $\tau_1\dots\tau_k$ arranged in increasing cyclic order along the boundary) is obtained by performing the relevant path integral while imposing the boundary conditions, which cut out a subset $\mathcal{C}$ in field configuration space:

$$\langle\mathcal{O}_{\alpha_k}(\tau_k)\dots\mathcal{O}_{\alpha_1}(\tau_1)\rangle_{\rm disk} = \tag{6.1}$$

$$= \int_{\mathcal{C}} \mathcal{D}[\phi]\mathcal{D}[\tilde{F}]\mathcal{D}[\eta]\mathcal{D}[\theta]\mathcal{D}[\rho]\, e^{-\tilde{S}_{\rm bulk}}\, {\rm Str}[\mathcal{O}_{\alpha_k}(\tau_k)U(\tau_k,\tau_{k-1})\ldots\mathcal{O}_{\alpha_1}(\tau_1)U(\tau_1,\tau_k)]\,.$$

In this section, we assume that $\mathcal{O}_{\alpha_j}$ are δ_b-closed:

$$\delta_b\mathcal{O}_{\alpha_j} = 0 \Longleftrightarrow \bar{D}_{\mathcal{B}}\alpha_j = 0\,. \tag{6.2}$$

6.1 Localization on B-model zero-modes

Remember that $\tilde{S}_{\rm bulk} = \tilde{S}_B + S_W$, where $\tilde{S}_B = \delta V_B$ is BRST exact. As in section 3, this allows us to replace $\tilde{S}_{\rm bulk}$ by $t\tilde{S}_B + S_W$, without changing the value of the correlator (6.1). Here t is a complex variable with positive real part. Invariance of (6.1) under changes in t follows by differentiation with respect to this parameter upon using δ_b-closure of $\mathcal{O}_{\alpha_j}$ and equation (5.11).[9]

We can now take the limit ${\rm Re}\, t \to +\infty$ to localize on the zero modes of $\tilde{S}_B$. This gives:

$$\langle \mathcal{O}_{\alpha_k}\ldots\mathcal{O}_{\alpha_1}\rangle_{\rm disk} = \langle\mathcal{O}_\alpha\rangle_{\rm disk} = N\int_{\mathcal{C}_0} d\phi d\eta d\theta e^{-\frac{A}{4}\tilde{L}_0}\, {\rm Str}[H_0\mathcal{O}_\alpha]\,. \tag{6.3}$$

To arrive at this formula, we noticed that the dependence of τ_j disappears on zero-modes, we set $\alpha := \alpha_k\wedge\cdots\wedge\alpha_1$ and integrated out the auxiliary fields $\tilde{F}$. Also notice that $\delta_b\mathcal{O}_\alpha = 0$ due to relations (6.2). The symbol H_0 denotes the restriction of the superholonomy factor H to zero-modes:

$$H_0 = e^{-\frac{il}{2}k_0}\,, \tag{6.4}$$

where l is the length of the disk's boundary and:

$$k_0 := \eta^{\bar{i}}\nabla_{\bar{i}}D^\dagger + [D, D^\dagger] = \delta_b D^\dagger\,. \tag{6.5}$$

The symbol $\mathcal{C}_0$ denotes the subset of the space of zero modes cut out by the boundary conditions. Since we integrated out the auxiliary fields, this subset is strictly BRST invariant:

$$\delta\mathcal{C}_0 \subset \mathcal{C}_0\,. \tag{6.6}$$

Using this property as well as δ-exactness (3.17) of $\tilde{L}_0$ and δ_b-exactness (6.5) of k_0, one checks[10] that (6.3) is insensitive to rescalings of these quantities, and hence can be replaced with:

$$\langle\mathcal{O}_\alpha\rangle_{\rm disk} = N\int_{\mathcal{C}_0} d\phi d\eta d\theta e^{-\lambda\tilde{L}_0}\, {\rm Str}[e^{-\mu k_0}\mathcal{O}_\alpha]\,, \tag{6.7}$$

where λ and μ are complex numbers such that ${\rm Re}\,\lambda > 0$. The quantity (6.7) is independent of the values of these two parameters.

To make (6.7) explicit, we must describe the restriction to $\mathcal{C}_0$. The relevant boundary condition takes the form (see appendix A):

$$i(G_{i\bar{j}}\eta^{\bar{j}} + \theta_i)\mathcal{U} = {\rm Str}[H(\tau)(\partial_i D + F_{i\bar{j}}\eta^{\bar{j}})] \Longleftrightarrow {\rm Str}[H(\tau)(\theta_i + iV_i)] = 0\,, \tag{6.8}$$

[9] The path integral over $\mathcal{C}$ of the BRST exact term involved in this argument vanishes because the boundary conditions determining $\mathcal{C}$ are preserved by the BRST transformations up to terms which vanish by the equations of motion for $\tilde{F}$ (see appendix A).

[10] The proof requires the identity $\delta\,{\rm Str}\,B = {\rm Str}\,\delta_b B$ for any quantity B built out of zero modes. This holds because the supertrace of any supercommutator vanishes.

where:

$$V_i := \partial_i D + (F_{i\bar{j}} - iG_{i\bar{j}})\eta^{\bar{j}} . \quad (6.9)$$

Equation (6.8) instructs us to replace θ_i by $-iV_i$ under the supertrace in order to produce the desired restriction. To implement these constraints, we shall use the quantity:

$$\Pi := \frac{1}{n!}\epsilon^{i_1 \dots i_n}(\theta_{i_1} 1_{\mathrm{End}(E)} + iV_{i_1}) \dots (\theta_{i_n} 1_{\mathrm{End}(E)} + iV_{i_n}) . \quad (6.10)$$

Consider an $\mathrm{End}(E)$-valued function f of θ_i:

$$f(\theta_1 \dots \theta_n) = \sum_{p=0}^{n} \sum_{1 \le i_1 < \dots < i_p \le n} \theta_{i_1} \dots \theta_{i_p} f^{i_1 \dots i_p} = \sum_{p=0}^{n} \frac{1}{p!}\theta_{i_1} \dots \theta_{i_p} f^{i_1 \dots i_p} , \quad (6.11)$$

where $f_{i_1 \dots i_p} \in \mathrm{End}(E)$ with $f^{i_{\sigma(1)} \dots i_{\sigma(p)}} = \epsilon(\sigma) f^{i_1 \dots i_p}$ for all $\sigma \in \Sigma_p$ and in the last equality we use implicit summation over $i_1 \dots i_p = 1 \dots n$. Here Σ_p is the group of permutations on p elements, while $\epsilon(\sigma)$ is the signature of the permutation σ. Then one checks the identity:

$$\int d\theta_1 \dots d\theta_n \Pi f(\theta_1 \dots \theta_n) = f(-iV_1 \dots - iV_n) , \quad (6.12)$$

where the right hand side is defined by:

$$\begin{aligned} f(-iV_1 \dots - iV_n) &:= \sum_{p=0}^{n} \sum_{1 \le i_1 < \dots < i_p \le n} \frac{1}{p!} \sum_{\sigma \in \Sigma_p} \epsilon(\sigma)(-iV_{i_{\sigma(1)}}) \dots (-iV_{i_{\sigma(p)}}) f^{i_1 \dots i_p} \\ &= \sum_{p=0}^{n} \frac{1}{p!}(-iV_{i_1}) \dots (-iV_{i_p}) f^{i_1 \dots i_p} . \end{aligned} \quad (6.13)$$

For later reference, we note the case $f = e^{q^i \theta_i}$, with q^i some Grassmann-odd quantities depending on ϕ and η. Then we have $f_{i_1 \dots i_p} = (-1)^{p(p+1)/2} q^{i_1} \dots q^{i_p}$ and find $f(-iV_1 \dots - iV_n) = e^{-iq^i V_i}$, using the fact that q_j are mutually anti-commuting. This gives:

$$\int d\theta_1 \dots d\theta_n e^{q^i \theta_i} \Pi = \int d\theta_1 \dots d\theta_n \Pi e^{q^i \theta_i} = e^{-iq^i V_i} . \quad (6.14)$$

Relation (6.12) shows that Π is a sort of 'Poincare dual' of $\mathcal{C}_0$ on the supermanifold of field configurations. Using (6.14), this observation allows us to write (6.7) as an unconstrained integral over the space of sphere zero-modes:

$$\langle \mathcal{O}_\alpha \rangle_{\mathrm{disk}} = N \int d\phi d\eta d\theta e^{-\lambda \tilde{L}_0}\, \mathrm{Str}[e^{-\mu k_0} \Pi \mathcal{O}_\alpha] . \quad (6.15)$$

Employing equation (6.14), we find:

$$\langle \mathcal{O}_\alpha \rangle_{\mathrm{disk}} = N \int d\phi d\eta\, \mathrm{Str}[e^{-\mu k_0} e^{-\lambda \tilde{L}_0^b} \mathcal{O}_\alpha] , \quad (6.16)$$

where:

$$\tilde{L}_0^b := \tilde{L}_0|_{\theta_i \to -iV_i} = -D_{\bar{i}} \partial_{\bar{j}} \bar{W} V^{\bar{i}} \eta^{\bar{j}} + G^{i\bar{j}}(\partial_i W)(\partial_{\bar{j}} \bar{W})$$

$$= \bar{H}^i{}_{\bar{j}}\eta^{\bar{j}}\partial_i D + \eta^{\bar{i}}\eta^{\bar{j}}\bar{H}^k_{\bar{i}}F_{k\bar{j}} + G^{i\bar{j}}(\partial_i W)(\partial_{\bar{j}}\bar{W})\,. \tag{6.17}$$

It is easy to check the relation:

$$\delta_b V_i = \partial_i W\,, \tag{6.18}$$

which shows that the on-shell BRST variation (3.16) of θ_i agrees with the δ_b-variation of $-iV_i$:

$$\delta_b(\theta_i 1_{\mathrm{End}(E)} + iV_i) = 0 \Longleftrightarrow (\delta\theta_i)1_{\mathrm{End}(E)} = -i\delta_b V_i \tag{6.19}$$

(this in particular implies that $\delta\Pi = 0$). Using this equation and relation (3.17), one finds that $\tilde{L}_0^b$ is δ_b-exact:

$$\tilde{L}_0^b = \delta_b \tilde{v}_0^b\,, \tag{6.20}$$

where:

$$\tilde{v}_0^b = \tilde{v}_0|_{\theta_i\to -iV_i} = V^{\bar{i}}\partial_{\bar{i}}\bar{W}\,. \tag{6.21}$$

Together with δ_b-exactness of k_0, this can be used to give a direct proof of independence of (6.16) of λ and μ.

We end by mentioning some useful properties of V_i. It is easy to compute the anti-commutator:

$$[V_i, V_j]_- = \partial_i\partial_j W - \delta_b[\partial_i\partial_j(D + A_{\bar{k}}\eta^{\bar{k}})]\,. \tag{6.22}$$

Moreover, it is not hard to check the identity:

$$[V_i, \mathcal{O}] = \partial_i(\delta_b\mathcal{O}) - \delta_b(\partial_i\mathcal{O}) \tag{6.23}$$

for any boundary observable $\mathcal{O}$ (as usual, the quantity on the left hand side is a supercommutator). In particular, a δ_b-closed boundary observable supercommutes with V_i up to a δ_b-exact term.

6.2 The space of boundary observables

After reduction to zero-modes, each boundary insertion $\mathcal{O}_\alpha$ can be identified with the superbundle-valued differential form α. This amounts to setting $\eta^{\bar{i}} \equiv dz^{\bar{i}}$, so the superalgebra $\mathcal{H}_b := \Omega^{(0,*)}(\mathrm{End}(E))$ provides an off-shell model for the space of boundary excitations. Moreover, equation (5.8) identifies the boundary BRST operator δ_b with the operator $\bar{D}_{\mathcal{B}} = \bar{\nabla}_A + \mathcal{D}$ acting on $\mathcal{H}_b$, where $\bar{\nabla}_A$ acts in the adjoint representation and $\mathcal{D} = [D,\cdot]$. The target space equations of motion imply that δ_b squares to zero. Thus $\mathcal{H}_b$ is a differential superalgebra. To be precise, we take $\mathcal{H}_b$ to consist of bundle-valued differential forms with at most polynomial growth at infinity.

The target space equations of motion imply the relations:

$$\bar{\partial}_A^2 = \mathcal{D}^2 = \bar{\partial}_A \circ \mathcal{D} + \mathcal{D}\circ\bar{\partial}_A = 0\,, \tag{6.24}$$

which show that $\mathcal{H}_b$ is a bicomplex. Thus the boundary BRST cohomology is computed by a spectral sequence E_*^b whose second term has the form:

$$E_2^b = H_{\mathcal{D}}(H_{\bar{\partial}_A}(\mathcal{H}_b))\,. \tag{6.25}$$

In the simple case $X = \mathbb{C}^n$, the holomorphic bundle E is the trivial superbundle of type (r_+, r_-) and the spectral sequence collapses to its second term. Then the BRST cohomology coincides with the cohomology of $\mathcal{D}$ taken in the space of square matrices of dimension $r_+ + r_-$ whose entries are polynomial functions of n complex variables. This recovers the result of [6].

6.3 The boundary-bulk and bulk-boundary maps

The equivalent expressions (6.15) and (6.16) allow us to extract an off-shell version of the boundary-bulk map of [28]:

$$f_\mu(\mathcal{O}_\alpha) = \mathrm{Str}[e^{-\mu k_0}\Pi\mathcal{O}_\alpha]\,. \tag{6.26}$$

This maps $\mathcal{H}_b$ to $\mathcal{H}$ and obeys:

$$\langle\mathcal{O}_\alpha\rangle_{\mathrm{disk}} = \langle f_\mu(\mathcal{O}_\alpha)\rangle_{\mathrm{sphere}} \Longleftrightarrow \mathrm{Tr}_b\,\alpha = \mathrm{Tr}\,f_\mu(\alpha)\,, \tag{6.27}$$

where we identified α with $\mathcal{O}_\alpha$. Here Tr and Tr_b are the bulk and boundary traces determined by the localization formulae (3.19) and (6.16):

$$\mathrm{Tr}\,\omega = \langle\mathcal{O}_\omega\rangle_{\mathrm{sphere}} = \int d\phi d\eta d\theta e^{-\lambda\tilde{L}_0}\mathcal{O}_\omega \tag{6.28}$$

$$\mathrm{Tr}_b\,\alpha = \langle\mathcal{O}_\alpha\rangle_{\mathrm{disk}} = \int d\phi d\eta\,\mathrm{Str}[e^{-\mu k_0}e^{-\lambda\tilde{L}_0^b}\mathcal{O}_\alpha]\,. \tag{6.29}$$

As in [28], we can also define a bulk-boundary map e through the adjunction formula:

$$\mathrm{Tr}(\mathcal{O}_\omega f_\mu(\mathcal{O}_\alpha)) = \mathrm{Tr}_b(e(\mathcal{O}_\omega)\mathcal{O}_\alpha)\,. \tag{6.30}$$

From the relations above, we find:

$$e(\mathcal{O}_\omega) := e^{\lambda\tilde{L}_0^b}\int d\theta_1\ldots d\theta_n e^{-\lambda\tilde{L}_0}\mathcal{O}_\omega\Pi\,. \tag{6.31}$$

This maps $\mathcal{H}$ to $\mathcal{H}_b$.

Using (6.19), we find that f_μ and e are compatible with the bulk and boundary BRST operators:

$$\delta\circ f_\mu = (-1)^n f_\mu\circ\delta_b \tag{6.32}$$

$$\delta_b\circ e = e\circ\delta\,. \tag{6.33}$$

To prove the second equation, we used the identity:

$$\delta\int d\theta_1\ldots d\theta_n f(\theta_1\ldots\theta_n) = \int d\theta_1\ldots d\theta_n\delta f(\theta_1\ldots\theta_n)\,, \tag{6.34}$$

which follows from (3.16). Relations (6.32) and (6.33) show that e and f_μ descend to well-defined maps e_* and f_* between the bulk and boundary BRST cohomologies (the latter are the maps considered in [28]). Since k_0 is δ_b-exact, one easily checks that f_* is independent of μ.

6.4 A geometric model for the boundary trace

As in section 3, we can use the identifications $\eta^{\bar{i}} \equiv dz^{\bar{i}}$ and $\theta_i \equiv \partial_i$ to represent our formulae in terms of standard geometric objects. We find:

$$\tilde{L}_0^b \equiv \bar{H} \lrcorner (\partial D + F) + ||\partial W||^2 = \delta_b \tilde{v}_0^b \tag{6.35}$$

with:

$$\tilde{v}_0^b = \mathrm{grad}\bar{W} \lrcorner \, (\partial D + F) \tag{6.36}$$

and:

$$k_0 = \bar{\nabla}_{\mathcal{B}} D^\dagger = \bar{\nabla}_A D^\dagger + [D, D^\dagger]\,. \tag{6.37}$$

The disk localization formula (6.16) becomes:

$$\mathrm{Tr}_b\, \alpha = N \int_X \Omega \wedge \mathrm{str}[e^{-\lambda \tilde{L}_0^b} \wedge e^{-\mu k_0} \wedge \alpha]\,, \tag{6.38}$$

while the quantity (6.10) takes the form:

$$\Pi = \frac{1}{n!}(\partial_1 + iV_1) \wedge \cdots \wedge (\partial_n + iV_n)\,, \tag{6.39}$$

with:

$$V_i = \partial_i D + (F_{i\bar{j}} - iG_{i\bar{j}})dz^{\bar{j}}\,. \tag{6.40}$$

Defining $V = dz^i \otimes V_i$, we obtain:

$$V = \partial D + F - iG\,, \tag{6.41}$$

where $G := G_{i\bar{j}} dz^i \otimes dz^{\bar{j}}$. Notice that here and above, ∂D is defined by $\partial D = dz^i \otimes \partial_i D$ (the order matters since D is odd).

6.5 Boundary localization pictures and the homotopy flow

As for the bulk sector, one can define a two-parameter semigroup of operators acting on $\mathcal{H}_b$ through:

$$U_b(\lambda, \mu)(\alpha) := e^{-\lambda \tilde{L}_0^b} \wedge e^{-\mu k_0} \wedge \alpha\,. \tag{6.42}$$

The pair (λ, μ) is taken inside the domain:

$$\Delta_b := \{(\lambda, \mu) \in \mathbb{C}^2 |\, \mathrm{Re}\, \lambda > 0\}\,. \tag{6.43}$$

Since both $\tilde{L}_0$ and $\tilde{L}_0^b$ are BRST-exact, each $U_b(\lambda, \mu)$ is homotopy-equivalent with the identity so this defines a homotopy flow. We let:

$$\mathrm{Tr}_b^B(\alpha) := N \int_X \Omega \wedge \mathrm{str}(\alpha) \tag{6.44}$$

denote the boundary trace of the B-twisted sigma model (viewed as a linear functional on the off-shell state space $\mathcal{H}_b$). Then equation (6.38) becomes:

$$\mathrm{Tr}_b(\alpha) = \mathrm{Tr}_b^B(U_b(\lambda, \mu)(\alpha))\,. \tag{6.45}$$

Again one can define localization pictures indexed by λ and μ. The boundary BRST operator satisfies:

$$U_b(\lambda, \mu) \circ \delta_b = \delta_b \circ U_b(\lambda, \mu)\,. \tag{6.46}$$

6.6 Residue formula for boundary correlators on the disk

As in section 3, we can use equation (6.15) to express boundary correlators in terms of generalized residues. For simplicity, we shall assume that the spectral sequence of Subsection 6.2 collapses to its second term. Setting $\mu = 0$ in (6.15) gives:

$$\langle \mathcal{O}_\alpha \rangle_{\text{disk}} = N \int d\phi d\eta d\theta e^{-\lambda \tilde{L}_0} \operatorname{Str}[\Pi \mathcal{O}_\alpha] \,. \tag{6.47}$$

We next take the limit $\lambda \to +\infty$ with $\lambda \in \mathbb{R}_+$. As in section 3, this forces the integral to localize on the critical points of W, while the gaussian approximation around these points becomes exact. Counting the powers of λ produced by the bosonic and fermionic gaussian integrals, we find that the correlator vanishes unless $\alpha = f$ with f a section of $\operatorname{End}(E)$. In this case, δ_b-closure of $\mathcal{O}_f$ amounts to the conditions $\bar{\nabla}_A f = 0$ and $[D, f] = 0$, and counting powers of λ shows that the only contributions which survive in the limit come from those pieces of the factor Π which are independent of θ and η. This gives:

$$\langle \mathcal{O}_\alpha \rangle_{\text{disk}} = 0 \qquad \text{unless } \alpha = f \in \operatorname{End}(E) \tag{6.48}$$

$$\langle \mathcal{O}_f \rangle_{\text{disk}} = \frac{C}{n!} \int_X \Omega \frac{\operatorname{Str}[(i\partial D)^{\wedge n} f(z)]}{\partial_1 W \ldots \partial_n W} \,, \tag{6.49}$$

where C is the constant introduced in section 3. These expressions generalize the residue formula proposed in [9]. Notice, however, that the residue formula of [9] arises for $\mu = 0$ and in the limit $\operatorname{Re} \lambda \to +\infty$. The limit proposed in [9] (namely $\operatorname{Re} \mu \to +\infty$ with $\lambda = 0$) does not suffice to localize the model's excitations unto the critical set of W.

7. Conclusions

We gave a detailed and general discussion of localization in the bulk and boundary sectors of B-type topological Landau-Ginzburg models. In the bulk sector, we showed that careful reconsideration of the localization argument of [1] leads to an entire family of localization formulae, parameterized by a complex number λ of positive real part. When real, this parameter measures the area of worldsheets with S^2 topology. The various "localization pictures" are related by a "homotopy flow" (a semigroup of operators homotopic to the identity), which implements rescalings of this area. The generalized localization argument leads to a one-parameter family of off-shell models for the bulk trace, extending the well-know result of [1]. The later is recovered for $\operatorname{Re} \lambda \to +\infty$, a degenerate limit which leads to the standard residue representation.

In the boundary sector, a similar argument gives a family of localization formulae parameterized by complex variables λ and μ subject to the condition $\operatorname{Re} \lambda > 0$. When real, these parameters describe the area of a worldsheet with disk topology, respectively the length of its boundary. The boundary localization pictures are once again related by a semigroup of homotopy equivalences, which implements rescaling of the disk's area and of the length of its boundary. This leads to a two-parameter family of off-shell models for the boundary trace. We also showed that the residue formula proposed in [9] arises in the limit $\lambda \to +\infty$ with $\mu = 0$, and generalizes to the set-up of [14, 12], which does not require constraints on the target space or on the rank of the holomorphic superbundle describing the relevant D-brane. In particular, this proves and generalizes the proposal of [9], though the residue representation we have found arises in a limit which differs from previous proposals. The argument required to establish this result is rather subtle, due to the complicated form of the boundary conditions.

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A. Euler-Lagrange variations and boundary conditions

Let us consider the Euler-Lagrange variations for our model. It is not hard to compute the variations of the bulk action:

$$\delta_\theta \tilde{S}_{\rm bulk} = \frac{i}{2}\int_\Sigma d^2\sigma\sqrt{g}\left[\varepsilon^{\alpha\beta}D_\beta\rho_\alpha^i + \frac{1}{2}G^{i\bar{k}}D_{\bar{k}}\partial_{\bar{j}}\bar{W}\eta^{\bar{j}}\right]\delta\theta_i \tag{A.1}$$

$$\delta_\eta \tilde{S}_{\rm bulk} = \frac{1}{2}\int_\Sigma d^2\sigma\sqrt{g}\left[g^{\alpha\beta}G_{\bar{i}j}D_\alpha\rho_\beta^j - \frac{i}{2}D_{\bar{i}}\partial_{\bar{j}}\bar{W}\theta^{\bar{j}}\right]\delta\eta^{\bar{i}} - \frac{1}{2}\int_{\partial\Sigma}d\tau G_{\bar{i}j}(\rho_n^j + i\rho_0^j)\delta\eta^{\bar{i}} \tag{A.2}$$

$$\begin{aligned}\delta_\rho \tilde{S}_{\rm bulk} &= \frac{1}{2}\int_\Sigma d^2\sigma\sqrt{g}\left[G_{i\bar{j}}(i\varepsilon^{\alpha\beta}D_\beta\theta^{\bar{j}} + g^{\alpha\beta}D_\beta\eta^{\bar{j}}) - \frac{1}{2}\varepsilon^{\alpha\beta}D_i\partial_j W\rho_\beta^j\right]\delta\rho_\alpha^i + \\ &\quad + \frac{i}{2}\int_{\partial\Sigma}d\tau G_{i\bar{j}}(\eta^{\bar{j}} + \theta^{\bar{j}})\delta\rho_0^i\end{aligned} \tag{A.3}$$

$$\begin{aligned}\delta_\phi \tilde{S}_{\rm bulk} &= \int_\Sigma d^2\sigma\sqrt{g}\left[-G_{\bar{i}j}\Delta\phi^j - \frac{i}{4}D_{\bar{i}}D_{\bar{j}}\partial_{\bar{k}}\bar{W}\theta^{\bar{j}}\eta^{\bar{k}} + \frac{i}{2}D_{\bar{i}}\partial_{\bar{j}}\bar{W}\tilde{F}^{\bar{j}}\right]\delta\phi^{\bar{i}} + \\ &\quad + \int_\Sigma d^2\sigma\sqrt{g}\left[-G_{i\bar{j}}\Delta\phi^{\bar{j}} - \frac{i}{2}D_i\partial_j W\tilde{F}^j + \frac{1}{8}\varepsilon^{\alpha\beta}D_iD_j\partial_k W\rho_\alpha^j\rho_\beta^k\right]\delta\phi^i + \\ &\quad + \int_{\partial\Sigma}d\tau G_{\bar{i}j}(\partial_n\phi^j + i\dot{\phi}^j)\delta\phi^{\bar{i}} + \int_{\partial\Sigma}d\tau G_{i\bar{j}}(\partial_n\phi^{\bar{j}} - i\dot{\phi}^{\bar{j}})\delta\phi^i\,.\end{aligned} \tag{A.4}$$

For the boundary coupling $\mathcal{U} = StrH(0)$, we find:

$$\delta\mathcal{U} = -\,\mathrm{Str}[H(0)I_C(\delta M)]\,, \tag{A.5}$$

where:

$$I_C(\delta M) = \int_0^l d\tau U(\tau)^{-1}\delta M(\tau)U(\tau)\,. \tag{A.6}$$

For δ_M we substitute the Euler-Lagrange variations:

$$\delta_\theta M = 0 \tag{A.7}$$

$$\delta_\eta M = -\left(\frac{1}{2}F_{\bar{i}j}\rho_0^j + \frac{i}{2}\nabla_{\bar{i}}D^\dagger\right)\delta\eta^{\bar{i}} \tag{A.8}$$

$$\delta_\rho M = -\frac{1}{2}(\partial_i D + F_{i\bar{j}}\eta^{\bar{j}})\delta\rho_0^i \tag{A.9}$$

and:

$$U^{-1}\delta_\phi M U = \frac{d}{d\tau}(U^{-1}A_{\bar{i}}\delta\phi^{\bar{i}}U) + U^{-1}(S_i\delta\phi^i + S_{\bar{i}}\delta\phi^{\bar{i}})U \tag{A.10}$$

with:

$$S_i = F_{i\bar{j}}\dot{\phi}^{\bar{j}} + \frac{1}{2}\partial_i F_{\bar{j}k}\eta^{\bar{j}}\rho_0^k + \frac{1}{2}\rho_0^j\partial_i\partial_j D + \frac{i}{2}\eta^{\bar{j}}[F_{i\bar{j}}, D^\dagger] + \frac{i}{2}[\partial_i D, D^\dagger] \tag{A.11}$$

$$S_{\bar{i}} = F_{\bar{i}j}\dot{\phi}^j + \frac{1}{2}\nabla_{\bar{i}} F_{\bar{j}k}\eta^{\bar{j}}\rho_0^k + \frac{1}{2}\rho_0^j[F_{\bar{i}j}, D] + \frac{i}{2}\eta^{\bar{j}}\nabla_{\bar{i}}\nabla_{\bar{j}}D^\dagger + \frac{i}{2}[D, \nabla_{\bar{i}}D^\dagger]\,. \tag{A.12}$$

This gives:

$$\delta_\theta\mathcal{U} = 0 \tag{A.13}$$

$$\delta_\eta\mathcal{U} = \int_{\partial\Sigma} d\tau\,\mathrm{Str}[H(\tau)(\frac{i}{2}\nabla_{\bar{i}}D^\dagger + \frac{1}{2}F_{\bar{i}j}\rho_0^j)]\delta\eta^{\bar{i}} \tag{A.14}$$

$$\delta_\rho\mathcal{U} = \frac{1}{2}\int_{\partial\Sigma} d\tau\,\mathrm{Str}[H(\tau)(\partial_i D + F_{i\bar{j}}\eta^{\bar{j}})]\delta\rho_0^i \tag{A.15}$$

$$\delta_\phi\mathcal{U} = -\int_{\partial\Sigma} d\tau\left(\mathrm{Str}[H(\tau)S_i]\delta\phi^i + \mathrm{Str}[H(\tau)S_{\bar{i}}]\delta\phi^{\bar{i}}\right). \tag{A.16}$$

To extract the boundary conditions, we write:

$$e^{-\tilde{S}_{\mathrm{bulk}}}\mathcal{U} = e^{-S_{\mathrm{eff}}}\,, \tag{A.17}$$

where $S_{\mathrm{eff}} = \tilde{S}_{\mathrm{bulk}} - \ln\mathcal{U}$ is viewed as a (non-local) worldsheet action. Since we desire local equations of motion, the boundary contributions to (A.1)–(A.4) must cancel the variation of $\ln\mathcal{U}$:

$$\mathcal{U}\delta\tilde{S}_{\mathrm{bulk}} = \delta\mathcal{U}\,. \tag{A.18}$$

Imposing this requirement, we find the boundary conditions:

$$G_{\bar{i}j}(\rho_n^j + i\rho_0^j)\mathcal{U} = -\,\mathrm{Str}\,H(\tau)(i\nabla_{\bar{i}}D^\dagger + F_{\bar{i}j}\rho_0^j) \tag{A.19}$$

$$i(G_{i\bar{j}}\eta^{\bar{j}} + \theta_i)\mathcal{U} = \mathrm{Str}\,H(\tau)(\partial_i D + F_{i\bar{j}}\eta^{\bar{j}}) \tag{A.20}$$

$$G_{\bar{i}j}(\partial_n\phi^j + i\dot{\phi}^j)\mathcal{U} = -\,\mathrm{Str}[H(\tau)S_{\bar{i}}] \tag{A.21}$$

$$G_{i\bar{j}}(\partial_n\phi^{\bar{j}} - i\dot{\phi}^{\bar{j}})\mathcal{U} = -\,\mathrm{Str}[H(\tau)S_i]\,. \tag{A.22}$$

The Euler-Lagrange equations can be read off from the bulk contributions to (A.1)–(A.4):

$$\epsilon^{\alpha\beta}D_\alpha\rho_\beta^i = \frac{1}{2}G^{i\bar{j}}D_{\bar{j}}\partial_{\bar{k}}\bar{W}\eta^{\bar{k}} \tag{A.23}$$

$$g^{\alpha\beta}D_\alpha\rho_\beta^i = \frac{i}{2}G^{i\bar{j}}D_{\bar{j}}\partial_{\bar{k}}\bar{W}\theta^{\bar{k}} \tag{A.24}$$

$$i\epsilon^{\alpha\beta}D_\beta\theta_i + g^{\alpha\beta}G_{i\bar{j}}D_\beta\eta^{\bar{j}} = \frac{1}{2}\epsilon^{\alpha\beta}D_i\partial_j W\rho_\beta^j\,. \tag{A.25}$$

It is not hard to see that the boundary conditions are BRST invariant modulo the equations of motion for F. For simplicity, we explain this for condition (A.20), which is of interest in section 6. Starting with (A.20), one easily computes the BRST variations of the left and right hand sides:

$$\delta(l.h.s.) = 2iG_{i\bar{j}}\tilde{F}^{\bar{j}}\mathcal{U} + i(G_{i\bar{j}}\eta^{\bar{j}} + \theta_i)\left[\frac{1}{2}\int_{\partial\Sigma}\rho^i\partial_i W\right]\mathcal{U} \tag{A.26}$$

$$\delta(r.h.s.) = (\partial_i W)\mathcal{U} + \left[\frac{1}{2}\int_{\partial\Sigma}\rho^k\partial_k W\right]\mathrm{Str}[H(\tau)(\partial_i D + F_{i\bar{j}}\eta^{\bar{j}})]\,. \qquad (A.27)$$

The two variations obviously agree if one uses equation (A.20), provided that the equation of motion $\tilde{F}^{\bar{i}} = -\frac{i}{2}G^{\bar{i}j}\partial_j W$ holds.

References

[1] C. Vafa, *Topological Landau-Ginzburg models*, *Mod. Phys. Lett.* **A 6** (1991) 337.

[2] E. Witten, *Phases of $N = 2$ theories in two dimensions*, *Nucl. Phys.* **B 403** (1993) 159 [hep-th/9301042].

[3] K. Hori, A. Iqbal and C. Vafa, *D-branes and mirror symmetry*, hep-th/0005247.

[4] N.P. Warner, *Supersymmetry in boundary integrable models*, *Nucl. Phys.* **B 450** (1995) 663 [hep-th/9506064].

[5] M. Kontsevich, *unpublished*.

[6] A. Kapustin and Y. Li, *D-branes in Landau-Ginzburg models and algebraic geometry*, *JHEP* **12** (2003) 005 [hep-th/0210296].

[7] D. Orlov, *Triangulated categories of singularities and D-branes in Landau-Ginzburg models*, math.AG/0302304.

[8] I. Brunner, M. Herbst, W. Lerche and B. Scheuner, *Landau-Ginzburg realization of open string TFT*, hep-th/0305133.

[9] A. Kapustin and Y. Li, *Topological correlators in Landau-Ginzburg models with boundaries*, *Adv. Theor. Math. Phys.* **7** (2004) 727 [hep-th/0305136].

[10] A. Kapustin and Y. Li, *D-branes in topological minimal models: the Landau-Ginzburg approach*, *JHEP* **07** (2004) 045 [hep-th/0306001].

[11] S.K. Ashok, E. Dell'Aquila and D.-E. Diaconescu, *Fractional branes in Landau-Ginzburg orbifolds*, hep-th/0401135.

[12] C.I. Lazaroiu, *On the boundary coupling of topological Landau-Ginzburg models*, hep-th/0312286.

[13] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley, New York 1978.

[14] J.M.F. Labastida and P.M. Llatas, *Topological matter in two-dimensions*, *Nucl. Phys.* **B 379** (1992) 220 [hep-th/9112051].

[15] E. Witten, *Chern-Simons gauge theory as a string theory*, *Prog. Math.* **133** (1995) 637–678 [hep-th/9207094].

[16] C.I. Lazaroiu, *Generalized complexes and string field theory*, *JHEP* **06** (2001) 052 [hep-th/0102122].

[17] C.I. Lazaroiu, *Unitarity, D-brane dynamics and D-brane categories*, *JHEP* **12** (2001) 031 [hep-th/0102183].

[18] C.I. Lazaroiu, *String field theory and brane superpotentials*, *JHEP* **10** (2001) 018 [hep-th/0107162].

[19] C.I. Lazaroiu, *D-brane categories*, *Int. J. Mod. Phys.* **A 18** (2003) 5299 [hep-th/0305095].

[20] C.I. Lazaroiu, *Graded lagrangians, exotic topological D-branes and enhanced triangulated categories*, *JHEP* **06** (2001) 064 [hep-th/0105063].

[21] C.I. Lazaroiu, R. Roiban and D. Vaman, *Graded Chern-Simons field theory and graded topological D-branes*, *JHEP* **04** (2002) 023 [hep-th/0107063].

[22] C.I. Lazaroiu and R. Roiban, *Holomorphic potentials for graded D-branes*, *JHEP* **02** (2002) 038 [hep-th/0110288].

[23] C.I. Lazaroiu, *An analytic torsion for graded D-branes*, *JHEP* **09** (2002) 023 [hep-th/0111239].

[24] C.I. Lazaroiu and R. Roiban, *Gauge-fixing, semiclassical approximation and potentials for graded Chern-Simons theories*, *JHEP* **03** (2002) 022 [hep-th/0112029].

[25] D.-E. Diaconescu, *Enhanced D-brane categories from string field theory*, *JHEP* **06** (2001) 016 [hep-th/0104200].

[26] G. Moore and G. Segal, unpublished; see http://online.kitp.ucsb.edu/online/mp01/.

[27] G.W. Moore, *Some comments on branes, G-flux and K-theory*, *Int. J. Mod. Phys.* **A 16** (2001) 936 [hep-th/0012007].

[28] C.I. Lazaroiu, *On the structure of open-closed topological field theory in two dimensions*, *Nucl. Phys.* **B 603** (2001) 497 [hep-th/0010269].

[29] E. Witten, *Mirror manifolds and topological field theory*, hep-th/9112056.

[30] M. Herbst, C.-I. Lazaroiu and W. Lerche, *Superpotentials, a(infinity) relations and wdvv equations for open topological strings*, *JHEP* **02** (2005) 071 [hep-th/0402110].

[31] D. Quillen, *Superconnections and the Chern character*, *Topology* **24** (1985) 89–95.